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Rapports de Recherche

N° 1071

Programme 3
Réseaux et Systèmes Répartis

MARTINGALES EXPONENTIELLES POUR LES FILES D'ATTENTE EN MILIEU ALEATOIRE : LE CAS M|G|1

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Août 1989

Juillet 1989

MARTINGALES EXPONENTIELLES POUR LES FILES D'ATTENTE EN MILIEU ALEATOIRE: LE CAS M | G | 1

par

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RESUME

Cet article est centré sur l'analyse de la file d'attente à un serveur avec des durées de services indépendantes et identiquement distribuées et un processus d'arrivée de Poisson dont l'intensité est modulée par un milieu aléatoire représenté par un processus de Markov à espace d'état fini. Des martingales exponentielles sont définies en liaison avec la chaîne de Markov du nombre des clients dans la file et de l'état du milieu aléatoire. Ces martingales fournissent une approche unifiée pour l'obtention de résultats connus sur la condition de stabilité et le régime stationnaire ainsi que plusieurs nouvelles propriétés sur la dynamique du système. Parmi ces nouveaux résultats, une loi de conservation reliant la durée de la période d'activité de la file à l'état du milieu à la fin de cette période, ainsi que des propriétés d'absolue continuité entre les lois de divers systèmes d'attente de ce type.

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July 1989

EXPONENTIAL MARTINGALES FOR QUEUES IN A RANDOM ENVIRONMENT: THE M | GI | 1 CASE

by

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ABSTRACT

This paper is concerned with single server queueing systems with renewal service process and Poisson arrivals modulated by a finite-state Markov chain. Exponential martingales are associated with a chain embedded at service completion epochs in the stochastic process describing the evolution of the number of customers in the queue and the state of the environment. The analysis of these martingale leads to a new and unified treatment of various known results concerning the stability condition and the steady state statistics, as well as to several new properties. Noteworthy among them are a conservation law that relates the duration of the busy period to the state of the environment at the end of the busy period, and some absolute continuity properties with respect to other queues of the same type.

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1. INTRODUCTION

The queues in random environment which have been considered in the literature have fallen into two basic categories, depending on whether the environment modulates the input stream or the server's speed. The present paper focuses on single server queueing systems where successive services are i.i.d. and independent of the arrivals which occur according to a Poisson process modulated by a finite-state Markov chain. The aim of the paper is to show that, as was the case for the $M|GI|1$ queue treated by the authors in [1,2], martingales again provide a unified tool for analyzing the dynamic, transient and stationary behaviors of these systems.

This is done by introducing several exponential martingales which are associated with a chain embedded at service completion epochs in the stochastic process describing the evolution of the number of customers in the queue and the state of the environment. The analysis of these martingales leads to a direct derivation of the stability condition for this class of $M|GI|1$ queues in random environment, and to various new conservation laws that relate the duration of the busy period to the state of the environment at the end of the busy period. These relations also take the form of a system of linear relations satisfied by the joint distributions of these random variables, so that this result can be combined with elements from the theory of Markov-renewal processes in order to work out the transient and stationary distributions.

The question of widening the range of validity of some of the conservation laws mentioned earlier, is investigated. To do so, it is neceesary to consider the absolute continuity properties of certain queues in random environment with respect to others. This arise naturally in the context of these martingale arguments, once it is recalled that a positive martingale can always be interpreted as a sequence of Radon-Nykodym derivatives (with respect to an underlying filtration).

As well known, the computation of the distributions encountered in such $M|GI|1$ queues in random environment leads unavoidably to complex analytical or algebraic manipulations. These questions are here approached within the martingale framework developed earlier. The discussion is only outlined, as the aim of the paper is more to establish a computational framework than to compute the distributions explicitly. This again illustrates the usefulness of martingale methods.

The paper is organized as follows: The model is described in detail in Section 2, together with the notation and assumptions used throughout the paper. A first set of martingales of interest is introduced in Section 3 and their use for studying system stability is demonstrated in Section 4, where results on the first passage time to the empty state and the conservation laws mentioned above are derived by direct probabilistic arguments. Section 5 is devoted to several analytical characterizations and to some computational issues. Several extensions of these results are then considered. The absolute continuity properties are treated in Section 6. More elaborate martingales are introduced in Section 7. Further conservation laws are derived yielding a new and unified approach for computing the stationary and transient statistics of the system.

2. THE MODEL - NOTATION AND ASSUMPTIONS

The collection of all integers (resp. non-negative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$), and let $\mathbb{R}$ (resp. $\mathbb{R}_+$) denote the set of all real (resp. non-negative real) numbers. The set of all complex numbers is denoted by $\mathbb{C}$. For any positive integer L, let $\mathbb{R}^{1\times L}$ (resp. $\mathbb{R}^{L\times 1}$) denote the space of L-dimensional row (resp. column) vectors with real entries, with a similar interpretation for the notation $\mathbb{R}_+^{1\times L}$ and $\mathbb{R}_+^{L\times 1}$. The $L \times L$ identity matrix is denoted by I_L.

The random variables (RVs) and stochastic elements considered in this paper are all defined on some fixed underlying probability triple $(\Omega, \mathcal{F}, P)$. Throughout, the characteristic function of any event A in $\mathcal{F}$ is denoted by $I[A]$. All continuous-time processes have $\mathbb{R}_+$ as time parameter set, and are assumed right-continuous with left limits.

2.1. The basic processes

The environment process is modeed as an irreducible Markov process $\{Y(t),\ t \geq 0\}$ taking values in a finite state space $\mathcal{L}$ which is represented for convenience by $\{1,\ldots,L\}$ for some positive integer L. This process is characterized by the $L \times L$ stochastic matrix $P := (P(i,j),\ 1 \leq i,j \leq L)$ of one-step transition probabilities for the embedded chain and by the row vector $\mu = (\mu(i),\ 1 \leq i \leq L)$ of rates out of the states $\{1,\ldots,L\}$.

The arrival pattern is Poissonian with intensity modulated by the environment in that $\lambda(i)$ is the intensity of arrivals when the environment is in state i, $1 \leq i \leq L$. Formally [3], the arrival process $\{A(t),\ t \geq 0\}$ is a counting process with $A(0) = 0$ such that the process $\{\alpha(t),\ t \geq 0\}$ defined by

$$\alpha(t) = A(t) - \int_0^t \lambda(Y(s-))ds, \quad t \geq 0 \tag{2.1}$$

is an $\mathcal{F}_t$-martingale where $\mathcal{F}_t = \sigma\{Y(s), A(s),\ 0 \leq s \leq t\}$.

The consecutive service times form a sequence of i.i.d. $\mathbb{R}_+$-valued RVs $\{S_n,\ n = 1,2,\ldots\}$ which is assumed independent of the environment process $\{Y(t),\ t \geq 0\}$ and of the arrival process $\{A(t),\ t \geq 0\}$. Throughout, the common probability distribution of the service times and its Laplace-Stieltjes transform are denoted by S and by S^*, respectively.

Finally, the initial queue size is modelled as an $\mathbb{N}$-valued RV Ξ which is independent of all other basic processes previously introduced.

2.2. The embedded queueing process

At time $t = 0$, a dummy customer is assumed to complete service and by leaving the system, it generates the 0^{th} departure. For $n = 0,1,\ldots$, let X_n° and Y_n° respectively denote the number of customers in the system and the state of the environment as seen by the n^{th} departing customer. For $n = 1,2,\ldots$, let $\overline{Y}_n^\circ$ represent the state of the environment at the beginning of the n^{th} service period while A_n° denotes the number of arrivals during the n^{th} service period. With these definitions, the queue size sequence $\{X_n^\circ,\ n = 0,1,\ldots\}$ satisfies the recursion

$$\begin{aligned} X_{n+1}^\circ &= X_n^\circ + A_{n+1}^\circ - I[X_n^\circ \neq 0] \\ X_0^\circ &= \Xi, \end{aligned} \qquad n = 0,1,\ldots \tag{2.2}$$

and the relations

$$\overline{Y}_{n+1}^\circ = Y_n^\circ \quad \text{if} \quad X_n^\circ \neq 0 \qquad n = 0,1,\ldots \tag{2.3}$$

hold true. Under the enforced assumptions, the $\mathbb{N} \times \mathcal{L}$-valued process $\{(X_n^\circ, Y_n^\circ),\ n = 0,1,\ldots\}$ is an *irreducible* Markov chain with a countable state space.

2.2. The free process

The process $\{X_n^\circ,\ n = 0,1,\ldots\}$ describes the actual evolution of the number of customers in the queue just after departure epochs, and can be viewed as the *reflected* version of a *free* "random walk" on $\mathbb{Z}$ with semi-Markov increments.

In order to define this free process, introduce the RVs $\{T_n,\ n = 0,1,\ldots\}$ naturally associated with $\{S_n,\ n = 1,2,\ldots\}$ by setting $T_0 = 0$ and

$$T_{n+1} = T_n + S_{n+1}, \qquad n = 0,1,\ldots \tag{2.4}$$

and define

$$A_{n+1} = A(T_{n+1}) - A(T_n) \qquad n = 0,1,\ldots \tag{2.5}$$

so that A_{n+1} represents the number of arrivals in the interval $(T_n, T_{n+1}]$.

The free process is described through the sequence of $\mathbb{Z} \times \mathcal{L}$-valued RVs $\{(X_n, Y_n),\ n = 0, 1, \ldots\}$, where

$$Y_n = Y(T_n), \qquad n = 0, 1, \ldots \quad (2.6)$$

and the $\mathbb{Z}$-valued sequence $\{X_n,\ n = 0, 1, \ldots\}$ satisfies the recursion

$$\begin{aligned} X_{n+1} &= X_n + A_{n+1} - 1 \\ X_0 &= \Xi. \end{aligned} \qquad n = 0, 1, \ldots \quad (2.7)$$

Under the enforced assumptions, the $\mathbb{Z} \times \mathcal{L}$-valued process $\{(X_n, Y_n),\ n = 0, 1, \ldots\}$ is an *irreducible* Markov chain with a countable state space.

The reflected and free processes are related to each other in the following sense: Let τ° denote the first passage time to the empty state for the queue size sequence $\{X_n^\circ,\ n = 0, 1, \ldots\}$ embedded at service completion epochs, i.e.,

$$\tau^\circ := \inf\{n \geq 0 : X_n^\circ = 0\} \qquad (2.8)$$

with the usual convention $\tau^\circ = \infty$ if the defining set in (2.8) is empty. Note that $\tau^\circ = 0$ if and only if $\Xi = 0$.

The RVs $\{A_{n+1}^\circ,\ n = 0, 1, \ldots\}$ and $\{A_{n+1},\ n = 0, 1, \ldots\}$ are not equal in general. Indeed, if the queueing system becomes empty for the first time as a result of the n^{th} service completion, i.e., $\tau^\circ = n$, then for all $k = 1, 2, \ldots$, the interval $(T_{n+k-1}, T_{n+k}]$ does not coincide anymore with the $(n+k)^{th}$ service period, and therefore $A_{n+k} \neq A_{n+k}^\circ$. On the other hand, it is plain from the definitions given above that for all $n = 0, 1, \ldots$,

$$X_n = X_n^\circ, \quad Y_n = Y_n^\circ \quad \text{and} \quad A_n = A_n^\circ \quad \text{on} \quad [\Xi \neq 0, n \leq \tau^\circ] \qquad (2.9)$$

In short, the processes $\{(X_n^\circ, Y_n^\circ),\ n = 0, 1, \ldots\}$ and $\{(X_n, Y_n),\ n = 0, 1, \ldots\}$ coincide up to time τ°.

2.4. The probabilistic building blocks

To fully characterize the probabilistic properties of the $M/GI/1$ queue in a random environment, it is convenient to introduce several additional quantities: The $L \times L$ *substochastic* matrices $T^z, 0 \leq z \leq 1$, have entries given by

$$T^z(i,j) := E[I[Y_1 = j] z^{A_1} \mid Y_0 = i] = \sum_{k=0}^{\infty} T_k(i,j) z^k, \quad 1 \leq i, j \leq L \qquad (2.10)$$

where for all k in $\mathbb{N}$, the coefficient $T_k(i,j)$ of z^k in (2.10) has the interpretation

$$T_k(i,j) := P[Y_1 = j, A_1 = k \mid Y_0 = i], \quad 1 \leq i, j \leq L. \qquad (2.11)$$

The radius of convergence of the matrix-valued mapping $z \to T^z$ is the scalar z^* defined by

$$z^* := \max_{1 \leq i,j \leq L} \sup\{z \geq 0 : \quad E[I[Y_1 = j] z^{A_1} \mid Y_0 = i] < \infty\}. \qquad (2.12)$$

The relation $1 \leq z^*$ is obvious. It follows from classical results on power series that the matrix-valued mapping $z \to T^z$ can be continued to complex values of the argument z provided $|z| < z^*$.

Let ν denote the first arrival date (or jump time) of the modulated Poisson process $\{A(t),\ t \geq 0\}$ after time $t = 0$, and let Q be the $L \times L$ stochastic matrix with entries given by

$$Q(i,j) := P[Y(\nu) = j \mid Y(0) = i], \quad 1 \leq i,j \leq L. \tag{2.11}$$

The next proposition summarizes the statistical properties of the sequences $\{(A_{n+1}, Y_n),\ n = 0,1,\ldots\}$ and $\{(A^\circ_{n+1}, Y^\circ_n),\ n = 0,1,\ldots\}$. For all $n = 0,1,\ldots$, let $\mathcal{F}_n$ and $\mathcal{F}^\circ_n$ denote the σ-fields of events generated by the RVs $\{\Xi, Y_0, (A_k, Y_k),\ \ 1 \leq k \leq n\}$ and $\{\Xi, Y_0, (A^\circ_k, Y^\circ_k),\ \ 1 \leq k \leq n\}$, respectively. It is plain that the RVs $\{X_k,\ k = 0,1,\ldots,n\}$ (resp. $\{X^\circ_k,\ k = 0,1,\ldots,n\}$) are all $\mathcal{F}_n$ (resp. $\mathcal{F}^\circ_n$)-measurable.

Proposition 2.1. *Under the foregoing assumptions, the relations*

$$P[A_{n+1} = k, Y_{n+1} = j \mid \mathcal{F}_n] = T_k(Y_n, j) \qquad n = 0,1,\ldots \tag{2.13}$$

and

$$\begin{aligned} &P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \mathcal{F}^\circ_n] \\ &= I[X^\circ_n \neq 0] T_k(Y^\circ_n, j) + I[X^\circ_n = 0] \sum_{\ell=1}^{L} Q(Y^\circ_n, \ell) T_k(\ell, j) \end{aligned} \qquad n = 0,1,\ldots \tag{2.14}$$

hold true for all (k,j) in $\mathbb{N} \times \mathcal{L}$.

Proof. The relation (2.13) is immediate in view of the enforced assumptions. To prove (2.14), recall (2.3) and observe that simple calculations and the strong Markov property readily show that for all (k,j) in $\mathbb{N} \times \mathcal{L}$,

$$\begin{aligned} &P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \mathcal{F}^\circ_n] \\ &= P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \mathcal{F}^\circ_n] I[X^\circ_n \neq 0] + P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \mathcal{F}^\circ_n] I[X^\circ_n = 0] \\ &= I[X^\circ_n \neq 0] T_k(Y^\circ_n, j) + I[X^\circ_n = 0] E[P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \sigma\{\overline{Y}^\circ_{n+1}\} \vee \mathcal{F}^\circ_n] \mid \mathcal{F}^\circ_n] \\ &= I[X^\circ_n \neq 0] T_k(Y^\circ_n, j) + I[X^\circ_n = 0] E[P[A^\circ_{n+1} = k, Y^\circ_{n+1} = j \mid \overline{Y}^\circ_{n+1}] \mid Y^\circ_n] \\ &= I[X^\circ_n \neq 0] T_k(Y^\circ_n, j) + I[X^\circ_n = 0] E[T_k(\overline{Y}^\circ_{n+1}, j) \mid Y^\circ_n]. \end{aligned} \qquad n = 0,1,\ldots \tag{2.15}$$

The result (2.14) now follows from (2.15) by making use of (2.11) together with the strong Markov property. □

Let z be a complex number such that $0 < |z| < z^*$. Upon multiplying both sides of (2.14) by z^k and summing up the resulting equations over k in $\mathbb{N}$, it readily follows that

$$\begin{aligned} &E[I[Y^\circ_{n+1} = j] z^{A^\circ_{n+1}} \mid \mathcal{F}^\circ_n] \\ &= I[X^\circ_n \neq 0] T^z(Y^\circ_n, j) + I[X^\circ_n = 0] \sum_{\ell=1}^{L} Q(Y^\circ_n, \ell) T^z(\ell, j) \\ &= I[X^\circ_n \neq 0] T^z(Y^\circ_n, j) + I[X^\circ_n = 0] (Q T^z)(Y^\circ_n, j) \end{aligned} \qquad n = 0,1,\ldots \tag{2.16}$$

for all $1 \leq j \leq L$. The integrability of the RV $I[Y^\circ_{n+1} = j] z^{A^\circ_{n+1}}$ follows from the definition of z^*, and implies the convergence of the sums $\sum_k T_k(Y^\circ_n, j) z^k$ and $\sum_k T_k(l, j) z^k$.

The form of the matrices T^z, $0 < |z| < z^*$, and Q is needed only at the end of the paper. Analytical expressions for these matrices are derived in Section 5 for later use. In due course, it will be established that Q is always invertible whereas T^z is "almost always" invertible, the necessary and sufficient condition for invertibility being given in Lemma 5.1 of Section 5.

3. THE MARTINGALES ASSOCIATED WITH THE FREE PROCESS

In this section, several sequences are shown to be $\mathcal{F}_n$-martingales; their usefulness is illustrated in the remainder of the paper.

For $0 < z < z^*$, the $\mathbb{R}^{1\times L}$-valued RVs $\{B_n^z,\ n = 0,1,\ldots\}$ are defined componentwise by

$$B_n^z(i) := I[Y_n = i]z^{X_n}, \quad 1 \le i \le L. \qquad n = 0,1,\ldots (3.1)$$

The RV B_n^z is $\mathcal{F}_n$-measurable.

The basic result of this section is now presented in the next proposition.

Proposition 3.1. *Fix z in $\mathbb{C}$ with $0 < |z| < z^*$. Under the foregoing assumptions, if the RV z^{Ξ} is integrable, then the RVs $\{B_n^z,\ n = 0,1,\ldots\}$ are all integrable and satisfy the relation*

$$E[B_{n+1}^z \mid \mathcal{F}_n] = \frac{1}{z}B_n^z T^z. \qquad n = 0,1,\ldots(3.2)$$

Proof. The integrability property is established by induction on n. It clearly holds for $n = 0$ by assumption. Assume it holds for some $n \ge 0$ and fix $1 \le j \le L$. It follows from the dynamics (2.2) and the definition (3.1) that

$$\begin{aligned}
E\left[|B_{n+1}^z(j)|\right] &= E\left[I[Y_{n+1} = j]|z|^{X_n+A_{n+1}-1}\right] \\
&= |z|^{-1}\sum_{i=1}^{L} E\left[I[Y_n = i]I[Y_{n+1} = j]|z|^{X_n+A_{n+1}}\right] \\
&= |z|^{-1}\sum_{i=1}^{L} E\left[B_n^{|z|}(i)E\left[I[Y_{n+1} = j]|z|^{A_{n+1}} \mid \mathcal{F}_n\right]\right] &(3.3)\\
&= |z|^{-1}\sum_{i=1}^{L} E\left[B_n^{|z|}(i)\right]T^{|z|}(i,j). &(3.4)
\end{aligned}$$

The passage to (3.3) is validated by the fact that the RVs X_n and Y_n are both $\mathcal{F}_n$-measurable, and (3.4) follows from (2.13). By the induction hypothesis, since $|B_n^z| = B_n^{|z|}$, it is now plain from (3.4) that the RV $B_{n+1}^z(j)$ is integrable, and this completes the induction step.

The arguments leading to (3.4) also show that

$$\begin{aligned}
E[B_{n+1}^z(j) \mid \mathcal{F}_n] &= E[I[Y_{n+1} = j]z^{X_n+A_{n+1}-1} \mid \mathcal{F}_n] \\
&= z^{X_n-1}E[I[Y_{n+1} = j]z^{A_{n+1}} \mid \mathcal{F}_n] \\
&= z^{X_n-1}T^z(Y_n,j) &n = 0,1,\ldots(3.5)
\end{aligned}$$

and the conclusion (3.2) is now immediate. □

3.1 The invertible case

The search for an exponential martingale is first carried out under the simplifying assumption that the matrix T^z is invertible for all z in $\mathbb{C}$, $0 < |z| < z^*$. In that case, let the $\mathcal{F}_n$-adapted sequence of $L \times L$ matrices $\{\Pi_n^z,\ n = 0,1,\ldots\}$ be defined by the relation

$$\Pi_n^z := z^n(T^z)^{-n}. \qquad n = 0,1,\ldots(3.6)$$

The next proposition identifies the martingale structure of a first sequence of RVs, in very much the same way as in the standard $M/GI/1$ situation treated by the authors [1].

Theorem 3.2. *Fix z in $\mathbb{C}$ with $0 < |z| < z^*$ and assume the RV z^Ξ to be integrable. If T^z is invertible, then the $\mathbb{R}^{1\times L}$- valued RVs $\{\tilde{M}_n^z,\ n = 0, 1, \ldots\}$ given by*

$$\tilde{M}_n^z := B_n^z \Pi_n^z \qquad n = 0, 1, \ldots (3.7)$$

form an integrable $\mathcal{F}_n$-martingale sequence.

Proof. The integrability property follows immediately from the integrability of the RVs $\{B_n^z,\ n = 0, 1, \ldots\}$ established in Proposition 3.1.

Since $\Pi_{n+1}^z = z(T^z)^{-1}\Pi_n^z$, relation (3.2) of Proposition 3.1 readily implies that

$$E[\tilde{M}_{n+1}^z \mid \mathcal{F}_n] = E[B_{n+1}^z \mid \mathcal{F}_n]\Pi_{n+1}^z = \frac{1}{z} B_n^z T^z \Pi_{n+1}^z = \tilde{M}_n^z. \qquad n = 0, 1, \ldots (3.8)$$

□

The results obtained so far are not satisfactory on at least two accounts. Indeed, the definition (3.6)-(3.7) depends in an essential way on the invertibility assumption made on the matrix T^z, and this very fact precludes the use of the martingale (3.7) for handling the general case. Moreover, as far as stability properties are concerned, the $M/GI/1$ queue in random environment is expected to behave as a one-dimensional system. It thus seems awkward, even when the matrix T^z is invertible, that the stability behavior should be studied through a higher-dimensional object. These remarks suggest that additional efforts be made to define a *one*-dimensional martingale under *no* assumption on the matrix T^z, that is structurally rich enough to provide information on system stability.

3.2. The general case

The invertibility assumption on the matrix T^z is now dropped. For every $0 < z < z^*$, the matrix T^z has *positive* coefficients and by virtue of the Perron-Frobenius Theorem [6], its eigenvalue of maximal norm – denoted by λ^z – is *real* and *strictly positive*. The corresponding left and right eigenvectors are the elements ψ^z and ϕ^z in $\mathbb{R}^{1\times L}$ and $\mathbb{R}^{L\times 1}$, respectively, which satisfy the equations

$$\psi^z T^z = \lambda^z \psi^z \quad \text{and} \quad T^z \phi^z = \lambda^z \phi^z. \qquad (3.9)$$

The components of the eigenvectors ψ^z and ϕ^z are all *strictly* positive. There is no loss of generality in assuming these vectors to be normalized in the sense that

$$\sum_{i=1}^{L} \phi^z(i) = 1 \quad \text{and} \quad \sum_{i=1}^{L} \psi^z(i) = 1. \qquad (3.10)$$

A careful examination of (3.2) suggests a natural way to define a one-dimensional martingale in the general case. Postmultiplication of (3.2) by the column vector ϕ^z and use of the eigenvector property (3.9) lead to the conclusion that

$$\begin{aligned} E[B_{n+1}^z \phi^z \mid \mathcal{F}_n] &= \frac{1}{z} B_n^z T^z \phi^z \\ &= \frac{\lambda^z}{z} B_n^z \phi^z = \frac{\lambda^z}{z} z^{X_n} \phi^z(Y_n). \end{aligned} \qquad n = 0, 1, \ldots (3.11)$$

On the other hand, it is plain that

$$B_{n+1}^z\phi^z = z^{X_{n+1}}\phi^z(Y_{n+1}) \qquad n = 0,1,\ldots \text{(3.12)}$$

and (3.11) can be rewritten as

$$E[z^{X_{n+1}}\phi^z(Y_{n+1}) \mid \mathcal{F}_n] = \frac{\lambda^z}{z} z^{X_n}\phi^z(Y_n). \qquad n = 0,1,\ldots \text{(3.13)}$$

This last relation suggests introducing the $\mathcal{F}_n$-adapted $\mathbb{R}$-valued RVs $\{M_n^z,\ n = 0,1,\ldots\}$ given by

$$M_n^z = z^{X_n}\phi^z(Y_n)\left[\frac{z}{\lambda}\right]^n \qquad n = 1,2,\ldots \text{(3.14)}$$

with

$$M_0^z = z^{X_0}\phi^z(Y_0). \qquad \text{(3.15)}$$

That the RVs $\{M_n^z,\ n = 0,1,\ldots\}$ are well defined is an easy consequence of the strict positivity properties of the eigenvalue λ^z stated earlier.

The next result parallels Theorem 3.2.

Theorem 3.3. *Fix z in $\mathbb{R}$ with $0 < z < z^*$. If the RV z^{Ξ} is integrable, then the $\mathbb{R}$-valued RVs $\{M_n^z,\ n = 0,1,\ldots\}$ form a positive integrable $\mathcal{F}_n$-martingale.*

Proof. Equations (3.13) and (3.14) give

$$\begin{aligned} E[M_{n+1}^z \mid \mathcal{F}_n] &= E[\phi^z(Y_{n+1})z^{X_{n+1}} \mid \mathcal{F}_n]\left[\frac{z}{\lambda}\right]^{n+1} \\ &= \phi_z(Y_n)z^{X_n}\left[\frac{z}{\lambda}\right]^n \\ &= M_n^z \end{aligned} \qquad n = 0,1,\ldots \text{(3.16)}$$

and the martingale property is thus established. □

This construction can be extended to complex values of the parameter z satisfying the constraint $0 < |z| < z^*$ and to all eigenpairs of the matrix T^z with a nonzero eigenvalue. Indeed, for $1 \le i \le L$ and $0 < |z| < z^*$. With $\lambda_i^z \neq 0$, let λ_i^z, ϕ_i^z and ψ_i^z respectively denote the i^{th} eigenvalue, right and left eigenvectors of T^z. With $\lambda_i^z \neq 0$, define the $\mathcal{F}_n$-adapted complex-valued RVs $\{M_n^{i,z},\ n = 0,1,\ldots\}$ by the relations

$$M_n^{i,z} = z^{X_n}\phi_i^z(Y_n)\left[\frac{z}{\lambda_i^z}\right]^n \qquad n = 1,2,\ldots\text{(3.17)}$$

with

$$M_0^{i,z} = z^{X_0}\phi_i^z(Y_0). \qquad \text{(3.18)}$$

The next result is an immediate extension of Theorem 3.3, and is established by the same arguments which are omitted for the sake of brevity.

Theorem 3.4. *Fix z in $\mathbb{C}$ with $0 < |z| < z^*$ and $\lambda_i^z \neq 0$ for $1 \le i \le L$. If the RV z^{Ξ} is integrable, then the $\mathbb{C}$-valued RVs $\{M_n^{i,z},\ n = 0,1,\ldots\}$ form an integrable $\mathcal{F}_n$-martingale.*

Proof. The integrability property follows from Proposition 3.1 while the martingale property is established as in Theorem 3.3. □

4. STABILITY RESULTS

Known stability results for the $M/GI/1$ queue in random environment are now derived by means of the martingale introduced in Section 3. Although these stability conditions are already available in the literature, this new derivation is nevertheless of interest on several accounts. Indeed, the proposed martingale arguments completely bypass the usual analysis of the invariant (or steady-state) distribution of the system state [7,9], and allow for the null and positive recurrent cases to be treated in a unified fashion. In addition, this new proof yields an interesting conservation law (given in (4.8) of Theorem 4.3) which relates the length of the busy period to the state of the environment at the end of the busy period. To the best of the authors' knowledge, this result appears to be new.

The following properties of the eigenvalue mapping $z \to \lambda^z$ are needed in the stability analysis.

Lemma 4.1. *The eigenvalue mapping $z \to \lambda^z$ is defined on the interval $(0, z^*) \cup (0,1]$ and takes values in $\mathbb{R}_+$; it is monotone increasing and convex with $0 < \lambda^z \leq 1$ for all z in $(0,1]$.*

Proof. Denoting transposition by ^{t}, Cayley's representation of the largest eigenvalue of a matrix implies

$$\lambda^z = \sup_{v \neq 0 \in \mathbb{R}^{1\times L}} \frac{vT^z v^t}{vv^t} = \sup_{v \neq 0 \in \mathbb{R}_+^{1\times L}} \frac{vT^z v^t}{vv^t} \tag{4.1}$$

where the second equality follows from the fact that the matrix T^z has positive entries. For each $v \neq 0$ in $\mathbb{R}_+^{1\times L}$, the mapping $z \to vT^z v^t (vv^t)^{-1}$ is obviously positive, monotone increasing and convex since each one of the coefficients of the matrix T^z has these three properties. A simple limiting argument based on (4.1) readily completes the proof. That $0 < \lambda^z \leq 1$ for all z in $(0,1]$ follows from the fact that T^z is substochastic on that range [6]. $\square$

The next lemma provides additional information on the eigenvalue mapping $z \to \lambda^z$. Note that the convexity of this mapping implies the existence of the left (resp. right) derivative $\frac{d^-}{dz}\lambda^z$ (resp. $\frac{d^+}{dz}\lambda^z$) over the interval $(0, z^*) \cup (0,1]$ (resp. $(0, z^* \vee 1)$). Moreover, $\frac{d^-}{dz}\lambda^z \neq \frac{d^+}{dz}\lambda^z$ in at most countably many points in the interval $(0, z^*)$. The matrix T^1, as defined in (2.4) for $z = 1$, is a stochastic matrix which is irreducible under the enforced assumptions. Hence, $\lambda^1 = 1$ and if π denotes the corresponding invariant measure, then π coincides with the left eigenvector ψ^1 of the matrix T^1.

Lemma 4.2. *With the notation given earlier, let ρ denote the constant defined by*

$$\rho := \sum_{i=1}^{L} \pi(i) E[A_1 \mid Y_0 = i]. \tag{4.2}$$

(i): The relation

$$\lim_{z \uparrow 1} \frac{d^-}{dz}\lambda^z = \frac{d}{dz}\lambda^z \,|_{z=1} = \rho \tag{4.3}$$

holds true,

(ii): If $\rho \leq 1$, then $z < \lambda^z$ for all $0 < z < 1$, and

(iii): If $\rho > 1$, then there exists ζ in the interval $(0,1)$ such that $\lambda^z < z$ for all $\zeta < z < 1$.

Proof. The vector ψ^z is a left eigenvector for the matrix T^z associated with the maximal eigenvalue λ^z. It then follows from (3.13) that $\psi^z T^z e = \lambda^z \psi^z e$, or equivalently that

$$\lambda^z = \frac{\psi^z T^z e}{\psi^z e} \tag{4.4}$$

where e denotes the vector in $\mathbb{R}^{L\times 1}$ with components all equal to 1.

Since the matrix-valued mapping $z \to T^z$ is analytic on $(0, z^*)$, it follows from standard arguments that the mappings $z \to \lambda^z$, $z \to \phi^z$ and $z \to \psi^z$ are all analytic on the interval $(0, z^*)$ except possibly for a finite number of isolated singularities. With this information in mind, differentiation of both members of (4.4) (when appropriate) yields

$$\dot{\lambda}^z = \frac{(\dot{\psi}^z T^z e + \psi^z \dot{T}^z e)(\psi^z e) - (\psi^z T^z e)(\dot{\psi}^z e)}{|\psi^z e|^2}, \tag{4.5}$$

where "$\cdot$" denotes differentiation with respect to the variable z. At $z = 1^-$, $T^z e = e$ since then T^z is a stochastic matrix, and $\psi^z = \pi$ so that $\psi^z e = \pi e = 1$. With these identifications and with the observation that $\psi^z \dot{T}^z e = \rho$ at $z = 1^-$, the relation (4.3) follows upon letting z go to 1 from below in (4.5).

Parts (ii) and (iii) are now straightforward consequences of (4.3) and of Lemma 4.1. □

The next result already demonstrates the power of the martingale properties discussed in the previous section. Let τ denote the first passage time to the empty state for the sequence $\{X_n,\ n = 0,1,\ldots\}$, i.e.,

$$\tau := \inf\{n \geq 0 : X_n = 0\} \tag{4.6}$$

with the usual convention $\tau = \infty$ if the defining set in (4.6) is empty. The first passage time τ is obviously an $\mathcal{F}_n$-stopping time, with

$$X_n \neq 0 \text{ whenever } 0 \leq n < \tau \tag{4.7a}$$

and

$$X_\tau = 0 \text{ on the event } [\tau < \infty]. \tag{4.7b}$$

It is noteworthy that $\tau = \tau^\circ$.

Theorem 4.3. *If $\rho \leq 1$, then the relation*

$$E\left[I[\tau < \infty]\phi^z(Y_\tau)\left[\frac{z}{\lambda^z}\right]^\tau \mid \mathcal{F}_0\right] = z^{X_0}\phi^z(Y_0) \tag{4.8}$$

holds for all $0 < z < 1$.

Proof. Owing to Theorem 3.3. and to Doob's Optional Sampling Theorem [8], the sequence of RVs $\{M^z_{\tau\wedge n},\ n = 0,1,\ldots\}$ is a $\mathbb{R}$-valued $\mathcal{F}_n$-martingale, with

$$M^z_{\tau\wedge n} = z^{X_{\tau\wedge n}}\phi^z(Y_{\tau\wedge n})\left[\frac{z}{\lambda^z}\right]^{\tau\wedge n}. \qquad n = 0,1,\ldots \tag{4.9}$$

The martingale property thus translate into the equalities

$$E\left[z^{X_{\tau\wedge n}}\phi^z(Y_{\tau\wedge n})\left[\frac{z}{\lambda^z}\right]^{\tau\wedge n} \mid \mathcal{F}_0\right] = z^{X_0}\phi^z(Y_0) \qquad n = 0,1,\ldots \tag{4.10}$$

valid for all $0 < z \leq 1$. A simple decomposition argument leads via (4.7) to a rewriting of (4.10) in the form

$$E\left[I[n < \tau]z^{X_n}\phi^z(Y_n)\left[\frac{z}{\lambda^z}\right]^n \mid \mathcal{F}_0\right] + E\left[I[\tau \leq n]\phi^z(Y_\tau)\left[\frac{z}{\lambda^z}\right]^\tau \mid \mathcal{F}_0\right]$$
$$= z^{X_0}\phi^z(Y_0). \qquad n = 0,1,\ldots \tag{4.11}$$

Whenever $0 < z < 1$, the bound

$$0 < z^{X_n}\phi^z(Y_n)\left[\frac{z}{\lambda^z}\right]^n \leq \left[\frac{z}{\lambda^z}\right]^n \max_{1\leq 1\leq L} \phi^z(i), \qquad n = 0,1,\ldots (4.12)$$

is obtained since $X_n \geq 0$ for $0 \leq n \leq \tau$. ¿From Lemma 4.2, $0 < \frac{z}{\lambda^z} < 1$ and therefore

$$\lim_n E\left[I[n < \tau]z^{X_n}\phi^z(Y_n)\left[\frac{z}{\lambda^z}\right]^n \mid \mathcal{F}_0\right] = 0. \tag{4.13}$$

by the Bounded Convergence Theorem for conditional expectations. On the other hand, the Monotone Convergence Theorem readily yields

$$\lim_n E\left[I[\tau \leq n]\phi^z(Y_\tau)\left[\frac{z}{\lambda^z}\right]^\tau \mid \mathcal{F}_0\right] = E\left[I[\tau < \infty]\phi^z(Y_\tau)\left[\frac{z}{\lambda^z}\right]^\tau \mid \mathcal{F}_0\right]. \tag{4.14}$$

Now upon taking the limit in (4.10) as n goes to ∞, (4.8) readily follows from (4.12) and (4.13). □

That Theorem 4.3 is indeed a statement on system stability is more apparent from the following corollary.

Corollary 4.4. *If $\rho \leq 1$, then*

$$P[\tau < \infty \mid \mathcal{F}_0] = 1 \quad P-a.s. \tag{4.15}$$

Proof. It is plain that $\lim_{z\uparrow 1} \lambda^z = 1$ and $\lim_{z\uparrow 1} \phi^z = e$, and the Bounded Convergence Theorem now yields the result upon letting z go to 1 from below in (4.8). □

The condition for system instability is now discussed.

Theorem 4.5. *If $\rho > 1$, then*

$$\lim_n X_n = \infty \quad P-a.s. \tag{4.16}$$

and

$$P[\tau < \infty \mid \mathcal{F}_0] < 1 \quad \text{on the event} \quad [\Xi \neq 0] \quad P-a.s. \tag{4.17}$$

Proof. With the notation $\gamma^z = \frac{\lambda^z}{z}$ for all $0 < z < 1$, the relation (3.13) takes the form

$$E[z^{X_{n+1}}\phi^z(Y_{n+1}) \mid \mathcal{F}_n] = \gamma^z z^{X_n}\phi^z(Y_n). \qquad n = 0,1,\ldots (4.18)$$

As pointed out in Lemma 4.2, there exists ζ in the interval $(0,1)$ such that $0 < \gamma^z < 1$ whenever $\zeta < z < 1$, in which case

$$E[z^{X_{n+1}}\phi^z(Y_{n+1}) \mid \mathcal{F}_n] \leq z^{X_n}\phi^z(Y_n). \qquad n = 0,1,\ldots (4.19)$$

In other words, the RVs $\{z^{X_n}\phi^z(Y_n),\ n = 0,1,\ldots\}$ form a bounded positive $\mathcal{F}_n$-submartingale and therefore converge both a.s. and in the mean [8, Thm. II-2-9, p. 26]. However, upon iterating (4.18), it is plain that

$$E[z^{X_{n+1}}\phi^z(Y_{n+1})] = |\gamma^z|^n\, E[z^{X_0}\phi^z(Y_0)], \qquad n = 0,1,\ldots (4.20)$$

so that for all $\zeta < z < 1$,

$$\lim_n z^{X_n} \phi^z(Y_n) = 0 \quad P - a.s. \tag{4.21}$$

necessarily. Consequently, $\lim_n z^{X_n} = 0$ $P - a.s.$ since the vector ϕ^z has all its components strictly positive, and the conclusion (4.16) follows immediately.

The proof of (4.17) follows an argument *ab absurdo.* To set the stage, assume that for *all* $x \neq 0$ in $\mathbb{N}$,

$$P[\tau < \infty \mid \Xi = x] = 1 \quad P - a.s. \tag{4.22}$$

and define the event Υ by

$$\Upsilon = [\Xi \neq 0] \cap [\lim_n X_n = \infty] \cap [\tau < \infty]. \tag{4.23}$$

It follows from (4.16) and (4.22) that

$$P[\Upsilon] = P[\Xi \neq 0]. \tag{4.24}$$

Consider the $\mathcal{F}_n$-stopping time σ defined by

$$\sigma := \inf\{n > \tau : X_n > 0\} \tag{4.25}$$

with the convention $\sigma = \infty$ if the defining set in (4.25) is empty. If it could be shown that $\sigma < \infty$ $P - a.s.$ on the event $[\Xi \neq 0]$, then the Markov property of the chain $\{(X_n, Y_n),\ n = 0, 1, \ldots\}$ would immediately imply the existence of an increasing family of $\mathcal{F}_n$-stopping times $\{\tau_k,\ k = 1, 2, \ldots\}$ such that a.s. *on* $[\Xi \neq 0]$, $\tau_k < \infty$ and $X_{\tau_k} = 0$ for all $k = 1, 2, \ldots$. As a result,

$$\underline{\lim}_n X_n \leq \lim_k X_{\tau_k} = 0 \quad \text{on the event} \quad [\Xi \neq 0] \quad P - a.s. \tag{4.26}$$

in clear contradiction with the convergence result (4.16). Consequently, the premise (4.22) has to be dismissed and (4.17) holds true.

In order to show that $\sigma < \infty$ $P - a.s.$ on the event $[\Xi \neq 0]$, fix a sample point ω in $[\tau < \infty]$. Owing to assumption (4.22), the process $\{X_n,\ n = 0, 1, \ldots\}$ starting in position $\Xi(\omega) \neq 0$ returns to the state 0 after a finite time $\tau(\omega)$. Then the convergence (4.16) guarantees that eventually the process becomes positive again in finite time, in fact for the first time at time $\sigma(\omega)$. The desired conclusion now follows by combining these remarks and (4.24). $\square$

The contents of Corollary 4.4 and Theorem 4.5 can be given the following more symmetric form.

Corollary 4.6. *Under the foregoing assumptions,*

$$P[\tau < \infty \mid \mathcal{F}_0] = 1 \quad \text{if and only if } \rho \leq 1. \tag{4.27}$$

Theorem 4.5 also admits the following immediate corollary.

Corollary 4.7. *If $\rho > 1$, then*

$$\lim_n X_n^\circ = \infty \quad P - a.s. \tag{4.28}$$

Proof. It is plain from Theorem 4.5 that

$$P[\lim_n X_n = \infty,\ \tau = \infty \mid \Xi \neq 0] > 0 \tag{4.29}$$

Since the free and reflected processes coincide up to time $\tau = \tau^\circ$ when $\Xi \neq 0$, (4.29) can be rewritten as

$$P[\lim_n X_n^\circ = \infty, \ \tau^\circ = \infty \mid \Xi \neq 0] > 0 \tag{4.30}$$

and therefore $\lim_n X_n^\circ = \infty$ with a positive probability. This and the fact that $\{(X_n^\circ, Y_n^\circ), \ n = 0, 1, \cdots\}$ is an irreducible Markov chain immediately imply $\lim_n X_n^\circ = \infty \ P - a.s.$ $\square$

These results can be combined into a necessary and sufficient condition of stability, which is similar to those already available in the literature [7,9]. To do so, define the average arrival rate $\overline{\lambda}$ by

$$\overline{\lambda} = \sum_{i=1}^{L} \pi(i)\lambda(i) \tag{4.31}$$

and set

$$E[S] = \int_0^\infty t dS(t) = E[S_n]. \qquad n = 1, 2, \ldots \tag{4.32}$$

Theorem 4.8. *Under the foregoing assumptions,*

$$\rho = \overline{\lambda} \cdot E[S]. \tag{4.33}$$

Proof. Define the filtration $\{\mathcal{H}_t, \ t \geq 0\}$ on Ω tby

$$\mathcal{H}_t := \sigma\{S_n, \ n = 1, 2, \ldots\} \bigvee \mathcal{F}_t, \quad t \geq 0 \tag{4.34}$$

Since the sequence of service times $\{S_n, \ n = 1, 2, \ldots\}$ is independent of the process $\{A(t), Y(t), \ t \geq 0\}$, it is plain that $T_1 (= S_1)$ is an $\mathcal{H}_t$-stopping time and that the process $\{\alpha(t), \ t \geq 0\}$ defined in (2.1) is also an $\mathcal{H}_t$-martingale. Consequently, with $A_1 = A(S_1)$,

$$\begin{aligned} E[A_1 \mid \mathcal{H}_0] =& E\left[\int_0^{S_1} \lambda(Y(t-))dt \mid \mathcal{H}_0\right] \\ =& E\left[\int_0^{S_1} \lambda(Y(t))dt \mid \mathcal{H}_0\right] \end{aligned} \tag{4.35}$$

where the last step follows from the right-continuity of paths. The assumed independence between the RV S_1 and the environment process $\{Y(t), \ t \geq 0\}$ implies

$$E\left[\int_0^{S_1} \lambda(Y(t))dt \mid \mathcal{H}_0\right] = \int_0^\infty I[S_1 > t] E[\lambda(Y(t)) \mid Y(0)] dt \tag{4.36}$$

since S_1 is $\mathcal{H}_0$-measurable.

As pointed out in Section 5, it is a simple matter to check that the probability vector π (which is invariant for the probability transition matrix T^1) is also invariant for the continuous-time process $\{Y(t), \ t \geq 0\}$. Let P_π denote any probability measure on the underlying sample space $(\Omega, \mathcal{F})$ that renders the process $\{Y(t), \ t \geq 0\}$ stationary, so that for all $t \geq 0$,

$$P_\pi[Y(t) = j | Y(0) = i] = \pi(j), \quad 1 \leq i, j \leq L. \tag{4.37}$$

In view of these last remarks, it is plain from (4.2) and (4.35)–(4.37) that

$$\begin{aligned}\rho &= E_\pi[A(S_1)] \\ &= \int_0^\infty P_\pi[S_1 > t]E_\pi[\lambda(Y(t))]dt \\ &= \overline{\lambda}\int_0^\infty P_\pi[S_1 > t]dt = \overline{\lambda}\cdot E_\pi[S_1] \end{aligned} \tag{4.38}$$

and the conclusion (4.33) follows. □

The formulae obtained so far used the eigenpair associated with the eigenvalue of maximal norm. It is plain from the discussion given at the end of Section 3 that these results can be extended to other eigenpairs. With the notation introduced there, the following strengthening of Theorem 4.3 holds.

Theorem 4.9. *For all* $1 \leq i \leq L$, *the relation*

$$E\left[I[\tau < \infty]\phi_i^z(Y_\tau)\left[\frac{z}{\lambda_i^z}\right]^\tau \mid (\mathcal{F})_0\right] = z^{X_0}\phi_i^z(Y_0) \tag{4.39}$$

holds for all z *in* $\mathbb{C}$ *with* $0 < |z| \leq 1$ *such that*

$$\lambda_i^z \neq 0 \quad \text{and} \quad \left|\frac{z}{\lambda_i^z}\right| < 1. \tag{4.40}$$

Proof. The proof is similar to the proof of Theorem 4.3. The arguments used for establishing (4.11) yield similarly

$$\begin{aligned} &E\left[I[n < \tau]z^{X_n}\phi_i^z(Y_n)\left[\frac{z}{\lambda_i^z}\right]^n \mid \mathcal{F}_0\right] + E\left[I[\tau \leq n]\phi_i^z(Y_\tau)\left[\frac{z}{\lambda_i^z}\right]^\tau \mid \mathcal{F}_0\right] \\ &= z^{X_0}\phi_i^z(Y_0). \qquad n = 0,1,\ldots \end{aligned} \tag{4.41}$$

The bound (4.12) is now replaced by

$$0 < |z|^{X_n}|\phi_i^z(Y_n)|\left|\frac{z}{\lambda^z}\right|^n \leq \max_{1\leq j\leq L} |\phi_i^z(j)|. \qquad n = 0,1,\ldots \tag{4.42}$$

and the remainder of the proof is exactly as in Theorem 4.3. □

Recall that $\tau^\circ = \tau$. Moreover the construction of the processes $\{(X_n^\circ, Y_n^\circ),\ n = 0,1,\ldots\}$ and $\{(X_n, Y_n),\ n = 0,1,\ldots\}$ implies that the stopped processes $\{(X_{\tau\wedge n}^\circ, Y_{\tau\wedge n}^\circ),\ n = 0,1,\ldots\}$ and $\{(X_{\tau\wedge n}, Y_{\tau\wedge n}),\ n = 0,1,\ldots\}$ coincide. Consequently, all the statements of this section, with particular attention to (4.8), (4.17), (4.27) and (4.39), hold for *both* the free and the reflected processes. For instance, (4.39) also reads

$$E\left[I[\tau^\circ < \infty]\phi_i^z(Y_{\tau^\circ}^\circ)\left[\frac{z}{\lambda_i^z}\right]^{\tau^\circ} \mid \mathcal{F}_0^\circ\right] = z^{X_0^\circ}\phi_i^z(Y_0^\circ). \tag{4.43}$$

5. ANALYTICAL CHARACTERIZATIONS

The first part of this section is devoted to the analytical characterization of the matrices Q and T^z. This characterization is then used to solve the functional equation (4.8) of Theorem 4.3 in order to determine the joint distribution of the busy period duration and the value of the environment process at the end of the busy period.

5.1. The matrix Q

Let Λ be the $L \times L$ diagonal matrix with entries given by

$$\Lambda(i,j) = \frac{\lambda(i)}{\lambda(i) + \mu(i)} \delta(i,j), \quad 1 \le i,j \le L. \tag{5.1}$$

¿From basic principles, it is clear that Q is given by

$$Q = \left[\sum_{m=0}^{\infty} ((I - \Lambda)P)^m \right] \Lambda = (I_L - (I_L - \Lambda)P)^{-1} \Lambda, \tag{5.2}$$

where the convergence of the Neuman series is a consequence of the fact that $(I_L - \Lambda)P$ is a submarkovian kernel. ¿From (5.2), it follows that Q is invertible with inverse given by

$$Q^{-1} = \Lambda^{-1}(I_L - (I_L - \Lambda)P). \tag{5.3}$$

5.2 The matrix T^z

Fix z in $(0,1)$. Recall that S_n is the duration of the n^{th} service, and denote by N_{n+1} the number of transitions of the environment process $\{Y(t),\ t \ge 0\}$ during the interval $(T_n, T_{n+1}]$, i.e., the $(n+1)^{rst}$ "service period" in the free system. With this notation, it is plain that

$$T^z = \int_0^\infty T_t^z \, dS(t), \tag{5.4}$$

where for each $t \ge 0$, the $L \times L$ matrix T_t^z has components given by

$$(T_t^z)(i,j) := E[I[Y_1 = j] z^{A_1} \mid Y_0 = i, S_1 = t], \quad 1 \le i,j \le L. \tag{5.5}$$

To proceed, consider the decomposition

$$T_t^z := \sum_{k=0}^{\infty} B_{k,t}^z, \quad t \ge 0 \tag{5.6}$$

where for each $t \ge 0$ and k in $\mathbb{N}$, $B_{k,t}^z$ is the $L \times L$ matrix with components given by

$$(B_{k,t}^z)(i,j) := E[I[Y_1 = j, N_1 = k] z^{A_1} \mid Y_0 = i, S_1 = t], \quad 1 \le i,j \le L. \tag{5.7}$$

Simple arguments show that the matrix $B_{0,t}^z$ is the $L \times L$ diagonal matrix given by

$$B_{0,t}^z(i,j) := e^{-\alpha^z(i)t} \delta(i,j), \quad 1 \le j \le L, \tag{5.8}$$

where

$$\alpha^z(i) := \mu(i) + \lambda(i)(1 - z), \quad 1 \le i \le L. \tag{5.9}$$

Furthermore, the recursions

$$B^z_{k+1,t}(i,j) = e^{-\alpha^z(i)t}\mu(i)\sum_{\ell=1}^{L} P(i,\ell)\int_0^t e^{\alpha^z(i)s}B^z_{k,s}(\ell,j)ds, \quad 1 \le i,j \le L, \qquad k = 0,1,\ldots \tag{5.10}$$

are readily obtained from basic principles and hold for all $t \ge 0$.

The notation is conveniently abbreviated by introducing the $L \times L$ diagonal matrices L^z_t and M with entries given by

$$L^z_t(i,j) := e^{\alpha^z(i)t}\delta(i,j), \quad 1 \le i,j \le L \tag{5.11}$$

and

$$M(i,j) := \mu(i)\delta(i,j), \quad 1 \le i,j \le L. \tag{5.12}$$

In matrix notation, (5.10) now reads

$$L^z_t B^z_{k+1,t} = \int_0^t M L^z_s P B^z_{k,s} ds, \quad t \ge 0 \tag{5.13}$$

so that the matrix function $t \to T^z_t$ satisfies the functional equation

$$L^z_t T^z_t = I_L + \int_0^t M L^z_s P T^z_s ds, \quad t \ge 0 \tag{5.14}$$

since the matrices L^z_t and $B^z_{0,t}$ are inverse of each other.

Differentiating (5.14) with respect to the time variable t now yields

$$\frac{\partial L^z_t}{\partial t} T^z_t + L^z_t \frac{\partial T^z_t}{\partial t} = M L^z_t P T^z_t, \quad t \ge 0 \tag{5.15}$$

or equivalently,

$$\frac{\partial T^z_t}{\partial t} = \left[MP - (L^z_t)^{-1}\frac{\partial L^z_t}{\partial t}\right] T^z_t, \quad t \ge 0 \tag{5.16}$$

after some simple rearrangements.

It is plain from (5.11) that

$$\frac{\partial L^z_t}{\partial t} = L^z_t A^z, \quad t \ge 0 \tag{5.17}$$

where A^z denotes the $L \times L$ diagonal matrix with entries given by

$$A^z(i,j) := \alpha^z(i)\delta(i,j), \quad 1 \le i,j \le L. \tag{5.18}$$

Direct substitution of (5.17) into (5.16) readily shows that the matrix function $t \to T^z_t$ satisfies the linear matrix differential equation

$$\frac{\partial T^z_t}{\partial t} = H^z T^z_t, \quad t \ge 0 \tag{5.19}$$

with initial condition $T^z_0 := I_L$, where H^z denotes the $L \times L$ matrix given by

$$H^z := MP - A^z. \tag{5.20}$$

The unique solution to (5.19) is known to be

$$T_t^z := e^{tH^z}, \quad t \geq 0 \tag{5.21}$$

so that for each $t \geq 0$, the matrix T_t^z is invertible, with inverse given by

$$[T_t^z]^{-1} = e^{-tH^z}, \quad t \geq 0. \tag{5.22}$$

For $1 \leq i \leq L$ and $0 < z < 1$, let β_i^z and $\tilde{\phi}_i^z$ (resp. $\tilde{\psi}_i^z$) denote the i^{th} eigenvalue and corresponding right (resp. left) eigenvector of the matrix H^z defined in (5.19). Moreover, recall that S^* denotes the Laplace transform of the service time distribution S.

Lemma 5.1. *The matrix T^z is given by*

$$T^z = \int_0^\infty e^{tH^z} dS(t), \tag{5.23}$$

For all $1 \leq i \leq L$, the complex number β_i^z has a negative real part so that $S^*(-\beta_i^z)$ is well defined. Moreover, the matrix T_z is invertible if and only if

$$\prod_{i=1}^{L} S^*(-\beta_i^z) \neq 0. \tag{5.24}$$

Proof. Equation (5.23) immediately follows from (5.4) and (5.22), and the rest of the proof deals only with the invertibility problem. Since the matrix H^z is a transient infinitesimal generator by construction, its eigenvalues β_i^z, $1 \leq i \leq L$, all have a negative real part and the matrix e^{tH^z} is thus a substochastic transition matrix. Its entries are therefore non-negative and bounded from above by 1, a fact which is sufficient to establish the existence of the integral in (5.23).

Let K^z be the Jordan decomposition of the matrix H^z [5, pp. 112 ff.]. Hence there exists an invertible matrix U^z with the property that

$$H^z = U^z K^z (U^z)^{-1}, \tag{5.25}$$

where K^z is the upper triangular matrix given by

$$K^z := \begin{pmatrix} \beta_1^z & a_2^z & 0 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2^z & a_3^z & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3^z & a_4^z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \beta_{L-2}^z & a_{L-1}^z & 0 \\ 0 & \cdots & 0 & 0 & 0 & \beta_{L-1}^z & a_L^z \\ 0 & \cdots & 0 & 0 & 0 & 0 & \beta_L^z \end{pmatrix} \tag{5.26}$$

In (5.26) the elements a_i^z, $2 \leq i \leq L$, are either zero or one, and the elements β_i^z, $1 \leq i \leq L$, are the eigenvalues of the matrix H^z counted with their multiplicities. Expanding (5.23) and using (5.25) lead to

$$T^z = U^z \left[\int_0^\infty \sum_{n=0}^{\infty} \frac{t^n}{n!} (K^z)^n dS(t) \right] (U^z)^{-1}. \tag{5.27}$$

The structure of (5.26) can now be used in (5.27) to claim the existence of an upper-triangular matrix O^z with i^{th} diagonal element $S^*(-\beta^z(i))$, $1 \leq i \leq L$, such that

$$T^z = U^z O^z (U^z)^{-1}. \tag{5.28}$$

The unspecified elements of the matrix O^z are necessarily finite owing to the remark made earlier on the convergence of the integral expression in (5.23). The matrix T^z is thus invertible if and only if none of the diagonal elements $S^*(-\beta^z(i))$, $1 \leq i \leq L$, are zero, and this completes the proof of the lemma. □

The case of a Laplace transform vanishing at some points of the complex right half-plane is rather infrequent. The coincidence of one of the complex eigenvalues of H_z with one of these possible zeros is an even more uncommon situation, which justifies the assertion of Section 2 that T^z is "almost always" invertible.

The next lemma establishes a property that is needed in the sequel, namely the link between the eigenpairs of T^z and H^z.

Lemma 5.2. *For all $1 \leq i \leq L$, $\lambda_i^z = S^*(-\beta_i^z)$ is an eigenvalue for the matrix T_z and the corresponding right (resp. left) eigenvector $\tilde{\phi}_i^z$ (resp. $\tilde{\psi}_i^z$) can be taken to be ϕ_i^z (resp. ψ_i^z).*

Proof. By (5.23) (and (5.27)), it is clear that

$$T^z = \int_0^\infty \sum_{n=0}^\infty \frac{(tH^z)^n}{n!} dS(t), \tag{5.29}$$

and for $1 \leq i \leq L$, the eigenpair property of $(\beta_i^z, \tilde{\phi}_i^z)$ for H^z reads

$$H^z \tilde{\phi}_i^z = \beta_i^z \tilde{\phi}_i^z. \tag{5.30}$$

Consequently,

$$\begin{aligned} T^z \tilde{\phi}_i^z &= \int_0^\infty \sum_{n=0}^\infty \frac{(tH^z)^n}{n!} \tilde{\phi}_i^z dS(t) \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{(t\beta_i^z)^n}{n!} \tilde{\phi}_i^z dS(t) \\ &= \int_0^\infty e^{t\beta_i^z} dS(t) \tilde{\phi}_i^z, \end{aligned}$$

i.e.,

$$T^z \tilde{\phi}_i^z = S^*(-\beta_i^z) \tilde{\phi}_i^z = \lambda_i^z \tilde{\phi}_i^z. \tag{5.31}$$

The proof is thus completed for the right eigenvector. The proof for the right eigenvector is similar, and is therefore omitted. □

5.3. Joint Distributions

As mentioned earlier, (4.8) provides a general relation between the length of the busy period of the queue and the state of the environment at the end of this busy period. It is now shown how this relation together with its extension given in Theorem 4.9 yields a simple way to determine analytically the joint distribution of these two quantities. The basic result in this direction is contained in the following technical lemma.

Lemma 5.3. *For every complex number $u \in \mathbb{C}$ such that $|u| < 1$, and for all $1 \leq i \leq L$, there exists a unique complex number $z_i(u)$ solution of the equation*

$$z = u\lambda_i^z. \tag{5.32}$$

Proof. Fix z such that $|z| = 1$ and $1 \leq i \leq L$. It is plain that $|S^*(-\beta_i^z)| \leq S^*(-\Re(\beta_i^z)) \leq 1$ where the last inequality follows from Lemma 5.1. Consequently,

$$|u\lambda_i^z| = |uS^*(-\beta_i^z) \leq |u| < |z| = 1$$

for every u in $\mathbb{C}$ such that $|u| < 1$. It follows from Rouché's Theorem that the function $z \to z - uS^*(-\beta_i^z)$ has exactly as many zeros in the unit disc as the function $z \to z$, namely one, and this completes the proof of the lemma. □

For $1 \leq j \leq L$ and u in $\mathbb{C}$ such that $0 < |u| < 1$, set

$$f(j,u) := E[I[\tau < \infty]\; I[Y_\tau = j]\; u^\tau \mid \mathcal{F}_0] = E[I[\tau^\circ < \infty]\; I[Y^\circ_{\tau^\circ} = j]\; u^{\tau^\circ} \mid \mathcal{F}^\circ_0]. \tag{5.33}$$

Theorem 5.4. *For all u in $\mathbb{C}$ such that $0 < |u| < 1$, the linear relation*

$$\sum_{j=1}^{L} f(j,u)\phi_i^{z_i(u)}(j) = z_i(u)^{X_0}\phi_i^{z_i(u)}(Y_0) \tag{5.34}$$

holds for all $1 \leq i \leq L$ such that $\lambda_i^{z_i(u)} \neq 0$.

Proof. The assumptions $u \neq 0$ and $\lambda_i^{z_i(u)} \neq 0$ entail $z_i(u) = u\lambda_i^{z_i(u)} \neq 0$. The martingale $\{M_n^{i,z_i(u)},\; n = 0,1,\ldots\}$ is thus well defined according to Theorem 3.4 and (5.34) is thus a mere rephrasing of (4.36) given in Theorem 4.8. □

In order to determine the joint distribution of interest, it suffices to determine the real numbers $f(j,u)$, $1 \leq i \leq L$, for $0 < u < 1$, or even in a real neighborhood of some real number $0 < u_0 < 1$. Theorem 5.4 shows that these real numbers satisfy a system of linear equations specified by (5.34), where the (possibly complex-valued) known parameters are the eigenvectors of the matrix T^z, or equivalently of matrix H^z, taken at $z = z_i(u)$. The natural question whether the rank of that system is sufficient to unambiguously determine these real numbers, is still open as to the writing of this paper.

6. ABSOLUTELY CONTINOUS FAMILIES OF QUEUES

The restrictions $0 < z < 1$ and $\rho \leq 1$ were essential in the proof of Theorem 4.3. What happens to the results of this theorem when the definitions are given for z lying in a larger set than the unit interval (in the event $1 < z^*$ and under appropriate integrability condition), or when $\rho > 1$? It is the purpose of this section to show that these questions can be settled by considering a larger family of queues in random environment, the paths of which are all absolutely continuous with respect to the paths of the initial queueing system.

The main results of the section are given in Theorems 6.1 and 6.3 and in Corollary 6.5 where a new conservation law generalizing (4.8) is established. The discussion is based on interpreting the real-valued martingale defined in Theorem 3.3. as a Radon-Nikodym derivative.

Theorem 6.1. *Fix z in the interval $(0, z^*) \cap (0, 1]$. Under the foregoing assumptions, the relation*

$$E\left[I[\tau < \infty]\phi^z(Y_\tau)\left[\frac{z}{\lambda^z}\right]^\tau \mid \mathcal{F}_0\right] = z^{X_0}\phi^z(Y_0) \tag{6.1}$$

holds true if and only if the condition

$$\psi^z \left[\frac{d}{dz} T^z\right] \phi^z \leq \frac{\lambda_z}{z} \tag{6.2}$$

is satisfied.

The proof of Theorem 6.1 proceeds in several steps which are organized in a series of technical lemmas. Observe that for $z = 1$, (6.2) reduces to the condition $\rho \leq 1$.

Lemma 6.2. *For all z in the interval $(0, z^*) \cup (0, 1]$ and $1 \leq i \leq L$, the mapping $\mathbb{N} \times \mathcal{L} \to \mathbb{R} : (k, j) \to \Theta_k^z(i, j)$ defined by*

$$\Theta_k^z(i,j) := \left[\frac{z^k \phi^z(j)}{\lambda^z \phi^z(i)}\right] T_k(i,j) \tag{6.3}$$

is a point mass probability function on the countable set $\mathbb{N} \times \mathcal{L}$.

Proof. Each one of the terms (6.3) is strictly positive, and for every pair (k, j) in $\mathbb{N} \times \mathcal{L}$,

$$\sum_{k=0}^{\infty} \sum_{j=1}^{L} \Theta_k^z(i,j) = \sum_{k=0}^{\infty} \sum_{j=1}^{L} T_k(i,j) \left[\frac{z^k \phi^z(j)}{\lambda^z \phi^z(i)}\right] \tag{6.4}$$

$$= \sum_{j=1}^{L} \frac{\phi^z(j)}{\lambda^z \phi^z(i)} \sum_{k=0}^{\infty} T_k(i,j) z^k \tag{6.5}$$

$$= \frac{1}{\lambda^z \phi^z(i)} \sum_{j=1}^{L} T^z(i,j) \phi^z(j) = 1, \tag{6.6}$$

where the second part of (6.6) follows from the eigenvalue property (3.13). □

At this point of the discussion it is convenient to introduce the set $\tilde{\Omega}$ defined as the cartesian product $(\mathbb{N} \times \mathcal{L})^\infty$, with generic element $\tilde{\omega}$ expressed in the form

$$\tilde{\omega} := (x_0, y_0, a_1, y_1, a_2, y_2, \ldots). \tag{6.7}$$

Let $\tilde{\Xi}$, $\{\tilde{A}_{n+1},\ n = 0, 1, \ldots\}$ and $\{\tilde{Y}_n,\ n = 0, 1, \ldots\}$ be the coordinate mappings on $\tilde{\Omega}$, i.e.,

$$\tilde{\Xi}(\tilde{\omega}) := x_0, \quad \tilde{A}_{n+1}(\tilde{\omega}) := a_{n+1} \quad \text{and} \quad \tilde{Y}_n(\tilde{\omega}) := y_n \qquad n = 0, 1, \ldots \tag{6.8}$$

with the representation (6.7). The filtration $\{\tilde{\mathcal{F}}_n,\ n = 0, 1, \ldots\}$ is defined on $\tilde{\Omega}$ by

$$\tilde{\mathcal{F}}_n := \sigma\{\tilde{\Xi}, \tilde{Y}_0, (\tilde{A}_k, \tilde{Y}_k), 0 < k \leq n\} \qquad n = 1, 2, \ldots \tag{6.9}$$

with $\tilde{\mathcal{F}}_0 := \sigma\{\tilde{\Xi}, \tilde{Y}_0\}$, and set $\tilde{\mathcal{F}}_\infty := \vee_n \tilde{\mathcal{F}}_n$ as usual.

Now, for every z in the interval $(0, z^*) \cup (0, 1]$, there exists a unique probability measure P^z on the σ-field $\tilde{\mathcal{F}}_\infty$ with the property that

$$P^z[\tilde{\Xi} = x, \tilde{Y}_0 = i] := P[\Xi = x, Y_0 = i] \tag{6.10a}$$

for all pairs (x, i) in $\mathbb{N} \times \mathcal{L}$, and

$$P^z[\tilde{A}_{n+1} = k, \tilde{Y}_{n+1} = j \mid \tilde{\mathcal{F}}_n] := \Theta_k^z(\tilde{Y}_n, j) \qquad n = 0, 1, \ldots \tag{6.10b}$$

for all (k,j) in $\mathbb{N} \times \mathcal{L}$.

The RVs $\{\tilde{X}_n, n = 0,1,\ldots\}$ are defined on $\tilde{\Omega}$ through the recursion

$$\begin{aligned}\tilde{X}_{n+1} &= \tilde{X}_n + \tilde{A}_{n+1} - I[\tilde{X}_n \neq 0], \\ \tilde{X}_0 &= \tilde{\Xi}.\end{aligned} \qquad n = 0,1,\ldots (6.11)$$

It is plain from (6.10)–(6.11) that under P^z, the RVs $\{(\tilde{X}_n, \tilde{Y}_n),\ n = 0,1,\ldots\}$ form a Markov chain with state space $\mathbb{N} \times \mathcal{L}$, and that

$$\tilde{\mathcal{F}}_n = \sigma\{(\tilde{X}_k, \tilde{Y}_k),\ 0 \leq k \leq n\}. \qquad n = 0,1,\ldots (6.12)$$

Upon specializing (6.3) for $z = 1$ and using the fact that $\lambda^z = 1$ and $\phi^z = e$ in that case, it is a simple matter to check that the stochastic process $\{(\tilde{X}_n, \tilde{Y}_n),\ n = 0,1,\ldots\}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_\infty, P^1)$ is equivalent in law to the original process $\{(X_n, Y_n),\ n = 0,1,\ldots\}$ on $(\Omega, \mathcal{F}, P)$.

In analogy with (4.6), the $\tilde{\mathcal{F}}_n$-stopping time $\tilde{\tau}$ is defined as

$$\tilde{\tau} := \inf\{n \geq 0 : \quad \tilde{X}_n = 0\}, \qquad (6.13)$$

with the usual convention $\tilde{\tau} = \infty$ whenever the defining set in (6.13) is empty. The following identification takes place.

Lemma 6.3. *Fix z in the interval $(0, z^*) \cup (0,1]$. the relation*

$$\begin{aligned}&E\left[I[n < \tau] z^{X_n} \phi_z(Y_n) \left[\frac{z}{\lambda^z}\right]^n \mid X_0 = x, Y_0 = i\right] \\ &= P^z[n < \tilde{\tau} \mid \tilde{X}_0 = x, \tilde{Y}_0 = i] z^x \phi_z(i)\end{aligned} \qquad n = 1,2,\ldots (6.14)$$

holds for every pair (x,i) in $\mathbb{N} \times \mathcal{L}$.

Proof. On the event $[n < \tau]$, (2.2) specializes to

$$X_k = X_0 + \sum_{\ell=1}^{k} A_\ell - k,\ 1 \leq k \leq n \qquad n = 1,2,\ldots (6.15)$$

so that

$$[n < \tau] = \bigcap_{k=0}^{n} [X_0 + \sum_{\ell=1}^{k} A_\ell > k] \qquad n = 1,2,\ldots (6.16)$$

and

$$\begin{aligned}&I[n < \tau] z^{X_n} \phi^z(Y_n) \left[\frac{z}{\lambda^z}\right]^n \\ &= I[X_0 + \sum_{\ell=1}^{k} A_\ell > k, 1 \leq k \leq n] \left[\frac{z^{X_0 + \sum_{k=1}^{n} A_k}}{(\lambda^z)^n}\right] \phi^z(Y_n).\end{aligned} \qquad n = 1,2,\ldots (6.17)$$

Now, in view of (6.15), define for every x in $\mathbb{N}$ the subset $K_n(x)$ of $\mathbb{N}^n$ by

$$K_n(x) := \{\vec{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n : x + \sum_{m=1}^{\ell} k_m > \ell, \quad 1 \leq \ell \leq n\}. \qquad n = 1,2,\ldots (6.18)$$

Fix a pair (x, i) in $\mathbb{N} \times \mathcal{L}$. For every $\vec{k}$ in $K_n(x)$ and every $\vec{\imath} = (i_1, ..., i_n)$ in $\mathcal{L}^n$ (with the convention $i_0 = i$), it is a simple matter to conclude by the Markov property that

$$\begin{aligned} &P[A_\ell = k_\ell, Y_\ell = i_\ell, 1 \le \ell \le n; X_0 = x, Y_0 = i] \\ &= T_{k_n}(i_{n-1}, i_n) P[A_\ell = k_\ell, Y_\ell = i_\ell, 1 \le \ell < n; X_0 = x, Y_0 = i], \quad n = 1,2,\ldots \end{aligned} \tag{6.19}$$

since $X_{n-1} \neq 0$ on the event $[X_0 = x, Y_0 = i; A_\ell = k_\ell, 1 \le \ell < n]$ for any $\vec{k}$ in $K_n(x)$. Iterating (6.19) readily yields

$$\begin{aligned} &P[A_\ell = k_\ell, Y_\ell = i_\ell, 1 \le \ell \le n; X_0 = x, Y_0 = i] \ \phi^z(i_n) \\ &= \left[\prod_{\ell=1}^{n} T_{k_\ell}(i_{\ell-1}, i_\ell) \frac{\phi^z(i_\ell)}{\phi^z(i_{\ell-1})}\right] P[X_0 = x, Y_0 = i] \ \phi^z(i_0) \\ &= \left[\prod_{\ell=1}^{n} \Theta^z_{k_\ell}(i_{\ell-1}, i_\ell)\right] \frac{(\lambda^z)^n}{z^{k_1+\ldots+k_n}} P[X_0 = x, Y_0 = i] \ \phi^z(i_0), \qquad n = 1,2,\ldots \end{aligned} \tag{6.20}$$

so that

$$\begin{aligned} &P[A_\ell = k_\ell, Y_\ell = i_\ell, 1 \le \ell \le n \mid X_0 = x, Y_0 = i] \ \phi^z(i_n) \\ &= \left[\prod_{\ell=1}^{n} \Theta^z_{k_\ell}(i_{\ell-1}, i_\ell)\right] \cdot \frac{(\lambda^z)^n}{z^{k_1+\ldots+k_n}} \phi^z(i). \qquad n = 1,2,\ldots \end{aligned} \tag{6.21}$$

The relations (6.17) and (6.21) now imply that

$$\begin{aligned} &E\left[I[n < \tau] z^{X_n} \phi^z(Y_n) \left[\frac{z}{\lambda^z}\right]^n \mid X_0 = x, Y_0 = i\right] \\ &= \sum_{\vec{k} \in K_n(x)} z^x \sum_{\vec{\imath} \in \mathcal{L}^n} \frac{z^{k_1+\ldots+k_n}}{(\lambda^z)^n} \phi^z(i_n) P[A_\ell = k_\ell, Y_\ell = i_\ell, 1 \le \ell \le n \mid X_0 = x, Y_0 = i] \\ &= \sum_{\vec{k} \in K_n(x)} \sum_{\vec{\imath} \in \mathcal{L}^n} \left[\prod_{\ell=1}^{n} \Theta^z_{k_\ell}(i_{\ell-1}, i_\ell)\right] z^x \phi^z(i) \\ &= P^z[\tilde{X}_\ell \neq 0, 1 \le \ell \le n \mid \tilde{X}_0 = x, \tilde{Y}_0 = i] \ z^x \phi^z(i) \\ &= P^z[n < \tilde{\tau} \mid \tilde{X}_0 = x, \tilde{Y}_0 = i] \ z^x \phi^z(i) \qquad n = 0,1,\ldots \end{aligned} \tag{6.22}$$

by straightforward arguments using the definition of the RVs τ and $\tilde{\tau}$, and the Markov property of the sequences $\{(X_n, Y_n),\ n = 0,1,\ldots\}$ and $\{(\tilde{X}_n, \tilde{Y}_n),\ n = 0,1,\ldots\}$. □

For z in the interval $(0, z^*) \cup (0, 1]$, define the $L \times L$ matrix Q^z with entries given by

$$Q^z(i,j) := \sum_{k=0}^{\infty} \Theta^z_k(i,j) = \frac{(T^z)(i,j)\phi^z(j)}{\lambda^z \phi^z(i)}, \quad 1 \le i,j \le L. \tag{6.23}$$

This matrix Q^z is a stochastic matrix as a consequence of (3.13).

Lemma 6.4. *For every z in the interval $(0, z^*) \cup (0,1]$, the stochastic matrix Q^z has an invariant measure γ^z given by*

$$\gamma^z(i) := \frac{\phi^z(i)\psi^z(i)}{\sum_{j=1}^{L} \phi^z(j)\psi^z(j)}, \quad 1 \le i \le L. \tag{6.24}$$

Proof. Indeed, since each one of the components of the vectors ϕ^z and ψ^z are strictly positive, so are the components of the vector γ^z. Moreover, for every $1 \leq j \leq L$,

$$\sum_{i=1}^{L} \phi^z(i)\psi^z(i)Q^z(i,j) = \sum_{i=1}^{L} \phi^z(i)\psi^z(i)\frac{\phi^z(j)}{\lambda^z\phi^z(i)}(T^z)(i,j)$$
$$= \left[\sum_{i=1}^{L} \psi^z(i)(T^z)(i,j)\right]\left[\frac{\phi^z(j)}{\lambda^z}\right] = \phi^z(j)\psi^z(j), \tag{6.25}$$

where the second equality in (6.25) follows from the eigenvalue property (3.13). □

All the elements are now present to give a discussion of Theorem 6.1.

Proof of Theorem 6.1. A closer inspection of the proof of Theorem 4.3 quickly reveals that on the interval $(0,z^*) \cup (0,1]$, under the extended conditions of Theorem 6.1, both (4.11) and (4.14) always hold, but the key limit (4.13) will fail to hold in general and the balance equation of Theorem 4.3 may not be valid. As a direct consequence of Lemma 6.3, the convergence

$$\lim_n E\left[I[n < \tau]z^{X_n}\left[\frac{z}{\lambda^z}\right]^n \phi_z(Y_n) \mid \mathcal{F}_0\right] = 0 \quad P-a.s. \tag{6.26}$$

takes place if and only if

$$\lim_n P^z[n < \tilde{\tau} \mid \tilde{\mathcal{F}}_0] = 0. \quad P^z - a.s. \tag{6.27}$$

By Theorem 4.6, applied to P^z in place of P, it follows that the convergence (6.27) takes place if and only if the appropriate version of the condition $\rho \leq 1$ holds. Upon making use of (4.2)–(4.3), it is a simple matter to check that the relevant condition is

$$\alpha^z = \sum_{i=1}^{L} \gamma_z(i) \sum_{k=0}^{\infty} k \sum_{j=1}^{L} \Theta_k^z(i,j) \leq 1. \tag{6.28}$$

To get the condition (6.2) from (6.28), observe that α^z can be evaluated as

$$\sum_{i=1}^{L} \phi^z(i)\psi^z(i) \sum_{j=1}^{L}\sum_{k=0}^{\infty}[z^k k\frac{\phi^z(j)}{\phi^z(i)}\frac{1}{\lambda^z}P[A_1 = k, Y_1 = j \mid \tilde{Y}_0 = i)]$$
$$= \sum_{i=1}^{L}\sum_{j=1}^{N} \psi^z(i)E[(A_1-1)z^{A_1}I[(Y_1 = j)] \mid \tilde{Y}_0 = i]\frac{\phi^z(j)}{\lambda^z}$$
$$= \frac{\psi^z(z\frac{dT^z}{dz})\phi^z}{\lambda^z}. \tag{6.29}$$

The end of the proof is now very similar to the proof of Theorem 4.3. First notice that for $0 < z < z^*$, the RVs

$$z^{X_{n\wedge\tau}}\left[\frac{z}{\lambda^z}\right]^{n\wedge\tau}\phi^z(Y_{n\wedge\tau}) \qquad n = 0,1,\ldots \tag{6.30}$$

also form an integrable $\mathcal{F}_n$-martingale owing to Theorem 3.3, so that (4.11) holds without modification for $0 < z < z^*$. □

The next corollary investigates the case where condition (6.2) is not satisfied.

Corollary 6.5. *Fix z in the interval $(0, z^*) \cup (0, 1]$. If*

$$\psi^z [\frac{d}{dz} T^z] \phi^z > \frac{\lambda_z}{z}, \tag{6.31}$$

then (6.1) has to be replaced by the relation

$$E\left[I[\tau < \infty] \phi^z(Y_\tau) \left[\frac{z}{\lambda^z} \right]^\tau \mid \mathcal{F}_0 \right] = z^{X_0} \phi^z(Y_0) - P^z[\tilde{\tau} = \infty \mid \tilde{\mathcal{F}}_0] \tag{6.32}$$

where P^z, $\tilde{\mathcal{F}}_0$ and $\tilde{\tau}$ are respectively defined by (6.10), (6.9) and (6.13).

Proof. If condition (6.2) is not satisfied, the limit given in (6.27) does not hold and has to be replaced by

$$\lim_n P^z[n < \tilde{\tau} \mid \mathcal{H}_0] = P^z[\tilde{\tau} = \infty \mid \mathcal{H}_0]. \tag{6.33}$$

This completes the proof of (6.32) when letting n go to ∞ in (4.11). □

7. THE MARTINGALES ASSOCIATED WITH THE REFLECTED PROCESS

In this section, several sequences associated with the reflected process $\{(X_n^\circ, Y_n^\circ),\ n = 0, 1, \ldots\}$ are shown to be $\mathcal{F}_n^\circ$-martingales. Their usefulness becomes apparent when analyzing other and more elaborate stopping times than the busy period, and when studying the transient and stationary statistics. Some of the proofs are very similar to those of Section 3 and are therefore omitted for the sake of brevity.

For $0 < z < z^*$, the $\mathbb{R}^{1 \times L}$-valued RVs $\{C_n^z,\ n = 0, 1, \ldots\}$ are defined componentwise by

$$C_n^z(i) := I[Y_n^\circ = i] z^{X_n^\circ}, \quad 1 \le i \le L \qquad n = 0, 1, \ldots \tag{7.1}$$

and the $L \times L$ matrices $\{S_n^z,\ n = 0, 1, \ldots\}$ are defined by

$$S_n^z := I[X_n^\circ \neq 0](\frac{1}{z}) I_L + I[X_n^\circ = 0] Q. \qquad n = 0, 1, \ldots \tag{7.2}$$

Both RVs C_n^z and S_n^z are $\mathcal{F}_n^\circ$-measurable.

Proposition 7.1. *Fix $0 < z < z^*$. Under the foregoing assumptions, if the RV z^Ξ is integrable, then the RVs $\{C_n^z,\ n = 0, 1, \ldots\}$ are all integrable and satisfy the relation*

$$E[C_{n+1}^z \mid \mathcal{F}_n^\circ] = C_n^z S_n^z T^z. \qquad n = 0, 1, \ldots \tag{7.3}$$

Proof. The integrability is handled as in Proposition 3.1. In order to get (7.3), observe that

$$\begin{aligned} E[C_{n+1}^z(j) \mid \mathcal{F}_n^\circ] &= E[I[Y_{n+1}^\circ = j] z^{X_n^\circ + A_{n+1}^\circ - I[X_n^\circ \neq 0]} \mid \mathcal{F}_n^\circ] \\ &= z^{X_n^\circ - I[X_n^\circ \neq 0]} E[I[Y_{n+1}^\circ = j] z^{A_{n+1}^\circ} \mid \mathcal{F}_n^\circ]. \qquad n = 0, 1, \ldots \end{aligned} \tag{7.4}$$

Substitution of (2.10) into (7.4) and use of the definition of the matrix S_n^z readily imply that

$$\begin{aligned} E[C_{n+1}^z(j) \mid \mathcal{F}_n^\circ] &= z^{X_n^\circ} \left\{ I[X_n^\circ \neq 0] \frac{1}{z} T^z(Y_n^\circ, j) + I[X_n^\circ = 0](QT^z)(Y_n^\circ, j) \right\} \\ &= z^{X_n^\circ} (S_n^z T^z)(Y_n^\circ, j). \qquad n = 0, 1, \ldots \end{aligned} \tag{7.5}$$

The conclusion (7.3) is now immediate from (7.5). □

7.1 The invertible case

Assume the matrix T^z to be invertible for $0 < z < z^*$. In that case, since the matrix Q is always invertible (see Section 5), the product matrix $S_n^z T^z$ is *invertible*, with corresponding inverse matrix R_n^z given by

$$R_n^z := (S_n^z T^z)^{-1} = I[X_n^\circ \neq 0] z (T^z)^{-1} + I[X_n^\circ = 0](QT^z)^{-1}. \qquad n = 0,1,\ldots (7.6)$$

The $\mathcal{F}_n^\circ$-measurable matrices $\{\Pi_n^z,\ n = -1, 0, \ldots\}$, $0 < z \le 1$, are now defined by

$$\Pi_n^z := \prod_{k=n}^{0} R_n^z = R_n^z \ldots R_1^z R_0^z, \qquad n = 0,1,\ldots (7.7)$$

with the convention $\Pi_{-1}^z = I_L$.

Theorem 7.2. *Fix $0 < z < z^*$ and assume the RV z^{Ξ} to be integrable. If T^z is invertible, the $\mathbb{R}^{1\times L}$- valued RVs $\{\tilde{N}_n^z,\ n = 0,1,\ldots\}$ given by*

$$\tilde{N}_n^z := C_n^z \Pi_{n-1}^z \qquad n = 0,1,\ldots (7.8)$$

form an integrable $\mathcal{F}_n^\circ$-martingale sequence.

Proof. It is easily seen by induction that each entry of the matrix Π_n^z is bounded from above and from below by some (non-random) constants. The stated integrability property now follows immediately from the integrability of the RVs $\{C_n^z,\ n = 0,1,\ldots\}$ established in Proposition 7.1.

The $\mathcal{F}_n^\circ$-measurability of the matrix Π_n^z and Proposition 7.1 readily imply that

$$E[\tilde{N}_{n+1}^z \mid \mathcal{F}_n^\circ] = E[C_{n+1}^z \mid \mathcal{F}_n^\circ]\Pi_n^z = C_n^z S_n^z T^z \Pi_n^z. \qquad n = 0,1,\ldots (7.9)$$

On the other hand, the very definition of the matrices R_n^z and Π_n^z yields the identity

$$S_n^z T^z \Pi_n^z = S_n^z T^z R_n^z \Pi_{n-1}^z = \Pi_{n-1}^z \qquad n = 0,1,\ldots (7.10)$$

and the martingale property is now immediate from (7.9) and (7.10). □

7.2. The general case

In the general case, postmultiplication of (7.3) by the column vector ϕ^z defined in Section 3.2 leads to the conclusion that

$$\begin{aligned} E[C_{n+1}^z \phi^z \mid \mathcal{F}_n^\circ] &= C_n^z S_n^z T^z \phi^z = \lambda^z C_n^z S_n^z \phi^z \\ &= \lambda^z z^{X_n^\circ} (S_n^z \phi^z)(Y_n^\circ). \end{aligned} \qquad n = 0,1,\ldots (7.11)$$

On the other hand, it is plain that

$$C_{n+1}^z \phi^z = z^{X_{n+1}^\circ} \phi^z(Y_{n+1}^\circ) \qquad n = 0,1,\ldots (7.12)$$

so that

$$\begin{aligned} E[z^{X_{n+1}^\circ} \phi^z(Y_{n+1}^\circ) \mid \mathcal{F}_n^\circ] &= \lambda^z z^{X_n^\circ} (S_n^z \phi^z)(Y_n^\circ) \\ &= \left[\frac{\lambda^z (S_n^z \phi^z)(Y_n^\circ)}{\phi^z(Y_n^\circ)}\right] z^{X_n^\circ} \phi^z(Y_n^\circ). \end{aligned} \qquad n = 0,1,\ldots (7.13)$$

This last relation suggests introducing the $\mathcal{F}_n^\circ$-adapted $\mathbb{R}$-valued RVs $\{N_n^z, \ n = 0,1,\ldots\}$ given by

$$N_n^z = z^{X_n^\circ}\phi^z(Y_n^\circ)\left[\prod_{k=0}^{n-1}\frac{\lambda^z(S_k^z\phi^z)(Y_k^\circ)}{\phi^z(Y_k^\circ)}\right]^{-1} \qquad n = 1,2,\ldots \quad (7.14)$$

with

$$N_0^z = z^{X_0^\circ}\phi^z(Y_0^\circ). \qquad (7.15)$$

That the RVs $\{N_n^z, \quad n = 0,1,\ldots\}$ are well defined is an easy consequence of the strict positivity properties of the eigenpair (λ^z,ϕ^z). It is also convenient at this point to observe that for all $1 \le i \le L$,

$$\frac{\lambda^z(S_n^z\phi^z)(i)}{\phi^z(i)} = I[X_n^\circ \neq 0]\frac{\lambda^z}{z} + I[X_n^\circ = 0]\left[\frac{\lambda^z(Q\phi^z)(i)}{\phi^z(i)}\right] \qquad (7.16)$$

$$= \left[\frac{\lambda^z}{z}\right]^{I[X_n^\circ\neq 0]}\left[\frac{\lambda^z(Q\phi^z)(i)}{\phi^z(i)}\right]^{I[X_n^\circ=0]} \qquad (7.17)$$

$$= \frac{\lambda^z}{z}\left[\frac{z(Q\phi^z)(i)}{\phi^z(i)}\right]^{I[X_n^\circ=0]} > 0 \qquad n = 0,1,\ldots \quad (7.18)$$

where the strict positivity is a consequence of the fact that Q is a stochastic matrix.

The next result parallels Theorem 7.2.

Theorem 7.3. *Fix $0 < z < z^*$. If the RV z^Ξ is integrable, then the $\mathbb{R}$-valued RVs $\{N_n^z, \ n = 0,1,\ldots\}$ form a positive integrable $\mathcal{F}_n^\circ$-martingale.*

Proof. For every $n = 0,1,\ldots$, the relation

$$\left[\prod_{0\le k<n}\frac{\lambda^z(S_k^z\phi^z)(Y_k^\circ)}{\phi^z(Y_k^\circ)}\right]^{-1} = \left[\frac{z}{\lambda^z}\right]^n\prod_{0\le k<n}\left[\frac{\phi^z(Y_k^\circ)}{z(Q\phi^z)(Y_k^\circ)}\right]^{I[X_k^\circ=0]} \qquad (7.19)$$

is an immediate consequence of (7.18), and the RVs $\{N_n^z, \ n = 0,1,\ldots\}$ thus take the alternate form

$$N_n^z = z^{X_n^\circ}\phi^z(Y_n^\circ)\left[\frac{z}{\lambda^z}\right]^n\prod_{0\le k<n}\left[\frac{\phi^z(Y_k^\circ)}{z(Q\phi^z)(Y_k^\circ)}\right]^{I[X_k^\circ=0]}. \qquad n = 0,1,\ldots \quad (7.20)$$

The matrix Q having all its entries positive, the bounds

$$0 < \frac{\phi^z(Y_n^\circ)}{z(Q\phi^z)(Y_n^\circ)} \le \frac{b^z}{z} \qquad n = 0,1,\ldots \quad (7.21)$$

are readily obtained, where the constant b^z is defined by

$$b^z = \max_{1\le i\le L}\left[\frac{\phi^z(i)}{(Q\phi^z)(i)}\right], \quad 0 < z < z^*. \qquad (7.22)$$

Consequently,

$$0 < N_n^z \le z^{X_n^\circ}\phi^z(Y_n^\circ)\left[\frac{z}{\lambda^z}\right]^n\left[\frac{b^z}{z}\right]^{\sum_{0\le k<n} I[X_k^\circ=0]} \qquad (7.23)$$

$$\le z^{X_n^\circ}\max_{1\le i\le L}\phi^z(i)\left[\frac{z}{\lambda^z}\right]^n\left[\max\{1,\frac{b^z}{z}\}\right]^n \qquad n = 0,1,\ldots \quad (7.24)$$

on the range $0 < z < z^*$, so that the integrability of the RVs $\{N_n^z,\ n = 0,1,\ldots\}$ immediately follows from the integrability of the RVs $\{z^{X_n^\circ},\ n = 0,1,\ldots\}$ which was established in the proof of Proposition 7.1.

The RVs

$$\frac{\lambda^z(S_k^z\phi^z)(Y_k^\circ)}{\phi^z(Y_k^\circ)}, \quad 0 \le k \le n, \tag{7.25}$$

being all $\mathcal{F}_n^\circ$-measurable, use of (7.13) and (7.20) leads to

$$E[N_{n+1}^z \mid \mathcal{F}_n^\circ] = E\left[\phi^z(Y_{n+1}^\circ)z^{X_{n+1}^\circ}\left[\prod_{k=0}^{n}\frac{\lambda^z(S_k^z\phi^z)(Y_k^\circ)}{\phi^z(Y_k^\circ)}\right]^{-1} \mid \mathcal{F}_n^\circ\right] \tag{7.26}$$

$$= \phi_z(Y_n^\circ)z^{X_n^\circ}\left[\prod_{k=0}^{n-1}\frac{\lambda^z(S_k^z\phi^z)(Y_k^\circ)}{\phi^z(Y_k^\circ)}\right]^{-1} = N_n^z \qquad n = 0,1,\ldots \tag{7.27}$$

and the martingale property is established. □

Several other conservation laws can be obtained from these new martingales. The simplest of these relations reads

$$E\left[\phi^z(Y_n^\circ)z^{X_n^\circ}\left[\frac{z}{\lambda^z}\right]^n \prod_{0\le k<n}\left[\frac{\phi^z(Y_k^\circ)}{z(Q\phi^z)(Y_k^\circ)}\right]^{I[X_k^\circ=0]}\right] = E[z^{X_0^\circ}\phi^z(Y_0^\circ)]$$

$$n = 0,1,\ldots \tag{7.28}$$

and follows immediately from the identity $E[N_n^z] = E[N_0^z]$ and from (7.19). The relation (7.28) establishes a general conservation between the state RVs $(X_n^\circ,\ Y_n^\circ)$ and the total time spent in an empty queue up to time n. Similar relations can be derived for certain stopping times such as the second time the queue empties or the first time a certain threshold is reached. This line of investigation will not be pursued here for the sake of brevity.

7.3. Stationary and transient distributions

The martingales introduced so far are now used to show how the computation of the stationary and transient distributions can in fact be obtained via basic theorems on Markov-renewal processes. This approach generalizes what was done in [2], where the authors computed the Pollaczek–Khinchine function and the transient distribution of the $M|GI|1$ queue via simple arguments from renewal theory.

Let σ be a finite $\mathcal{F}_n^\circ$-stopping time and let $\nu(\sigma)$ be the $\mathcal{F}_n^\circ$-stopping time

$$\nu(\sigma) := \inf\{n > \sigma : X_n^\circ = 0\}$$

with the usual convention $\nu(\sigma) = \infty$ if the defining set is empty.

Theorem 7.4 *If* $\rho \le 1$, *the relation*

$$E[I[\sigma < \infty, \nu(\sigma) < \infty]\left[\frac{z}{\lambda^z}\right]^{\nu(\sigma)-\sigma}\phi^z(Y_{\nu(\sigma)}^\circ) \mid \mathcal{F}_\sigma^\circ] = I[\sigma < \infty]z^{X_\sigma^\circ}\phi^z(Y_\sigma^\circ)\left[\frac{z(Q\phi^z)(Y_\sigma^\circ)}{\phi^z(Y_\sigma^\circ)}\right]^{I[X_\sigma^\circ=0]}$$

$$\mathrm{P-a.s.} \tag{7.29}$$

holds for all $0 < z < 1$.

Proof. In view of the strong Markov property and the time homogeneity, it suffices to establish (7.29) for $\sigma = 0$. To that end apply the Optional Sampling Theorem [8] to the martingale $\{N^z(n),\ n = 0,1,\ldots\}$ defined in (7.14) with the stopping times 0 and $\nu(0) \wedge n$. The relation

$$E\left[I[n < \nu(0)]z^{X_n^\circ}\phi^z(Y_n^\circ)\left[\frac{z}{\lambda^z}\right]^n \mid \mathcal{F}_0^\circ\right] + E\left[I[\nu(0) \le n]\phi^z(Y_{\nu(0)}^\circ)\left[\frac{z}{\lambda^z}\right]^{\nu(0)} \mid \mathcal{F}_0^\circ\right]$$
$$= z^{X_0^\circ}\phi^z(Y_0^\circ)\left[\frac{z(Q\phi^z)(Y_0^\circ)}{\phi^z(Y_0^\circ)}\right]^{I[X_0^\circ=0]} \qquad n = 0,1,\ldots (7.30)$$

is then seen to hold, and (7.29) follows from (7.30) upon using the same limiting arguments as in the proof of Theorem 4.3. □

Let $\{\nu_n,\ n = 0,1,\ldots\}$ be the sequence of $\mathcal{F}_n^\circ$-stopping times defined by the recursion

$$\nu_{n+1} = \nu(\nu_n) \qquad n = 0,1,\ldots (7.31)$$

with $\nu_0 = 0$. With $\sigma = 0$, the arguments of Corollary 4.4 applied to (7.29) imply

$$P[\nu_1 < \infty \mid \mathcal{F}_0^\circ] = 1 \quad \mathrm{P-a.s.} \qquad (7.32)$$

Lemma 7.5. *If $\rho \le 1$, the RVs $\{(Y_{\nu_n}^\circ,\ \nu_n),\ n = 1,2,\ldots\}$ form a (possibly delayed) recurrent Markov-renewal process.*

Proof. It is plain from the Markov property that for all $n = 0,1,\ldots$, the RVs $\{Y_{\nu(n+1)}^\circ,\ \nu_{n+1} - \nu_n\}$ and the σ-field $\mathcal{F}_{\nu_n}^\circ$ are conditionally independent given the RV $Y_{\nu_n}^\circ$. This completes the proof of the Markov-renewal property. Moreover this Markov-renewal process will be delayed if and only if $\Xi = 0$ $P-a.s.$ For $\rho \le 1$, the recurrence property is an immediate consequence of (7.32). □

The forward recurrence times $\{\tau(n),\ n = 0,1,\ldots\}$ of the recurrent renewal process $\{\nu_n,\ n = 0,1,2,\ldots\}$ are defined by

$$\tau(n) = \begin{cases} \inf\{m \ge 0 : X_{n+m}^\circ = 0\} & \text{if this set is non empty ;} \\ \infty & \text{otherwise .}\end{cases} \qquad n = 0,1,\ldots (7.33)$$

The generating function of the number X_n° of customers at the n^{th} service completion is related to the generating function of the forward recurrence time $\tau(n)$ in a very simple way. This key relationship is provided in the next theorem.

Theorem 7.6. *Assume $\rho \le 1$. For all $0 < z \le 1$, the relation*

$$E\left[z^{X_n^\circ}\phi_i^z(Y_n^\circ)\right] = E\left[\left[\frac{z}{\lambda_i^z}\right]^{\tau(n)}\phi_i^z(Y_{n+\tau(n)}^\circ)\right] \qquad n = 0,1,\ldots (7.34)$$

holds true for all $1 \le i \le L$ such that $\lambda_i^z \neq 0$ and $|\frac{z}{\lambda_i^z}| < 1$.

Proof. In view of the homogeneity property, the martingale relations (4.43) readily yield the identity

$$E\left[I[\tau(n) < \infty]\phi_i^z(Y_{n+\tau(n)}^\circ)\left[\frac{z}{\lambda_i^z}\right]^{\tau(n)} \mid \mathcal{F}_n^\circ\right] = z^{X_n^\circ}\phi_i^z(Y_n^\circ) \qquad (7.35)$$

whenever $\lambda_i^z \neq 0$, $1 \le i \le L$, and the relation (7.34) follows by taking expectations on both sides of (7.35). □

Various generating functions pertaining to the transient and stationary distributions of the queue size process can now receive a very simple interpretation in terms of the generating function of the forward recurrence times defined in (7.33). To that end, for every z in $\mathbb{R}$ such that $0 < z < 1$, set

$$g_n(j,z) := E[I[Y_n^\circ = j]\, z^{X_n^\circ}], \quad 1 \le j \le L. \qquad n = 0,1,\ldots (7.36)$$

With this notation, Theorem 7.6 can be rephrased as follows.

Corollary 7.7. *For all z in $\mathbb{R}$ such that $0 < z < 1$, the linear relation*

$$\sum_{j=1}^{L} g_n(j,z)\phi_i^z(j) = E\left[\left[\frac{z}{\lambda_i^z}\right]^{\tau(n)} \phi_i^z(Y^\circ_{n+\tau(n)})\right] \qquad n = 0,1,\ldots (7.37)$$

holds for all $1 \le i \le L$ such that $\lambda_i^z \neq 0$ and $|\frac{z}{\lambda_i^z}| < 1$.

In other words, the generating functions of interest (7.36) – be it in the transient or stationary regime – satisfy the system (7.37) of linear equations where the (possibly complex-valued) known parameters of the left handside are the eigenvectors of the matrix T^z, or equivalently of the matrix H^z, and those in the right handside are given by the statistics of the Markov-renewal process defined in Lemma 7.5.

7.4. Remarks on the system (7.37)

Although it is beyond the scope of this paper to provide a complete analytical characterization of the generating functions (7.36), some brief comments are in order concerning the computational issues associated with the linear system of Corollary 7.7.

(i): Computation of the right handside of (7.37): The first question concerns the computation of the stationary or transient statistics of the Markov-renewal process showing up in the rigth handside of (7.37). This can be obtained within the existing martingale framework as follows. Indeed the martingale which was defined in (7.14) – upon applying the Optional Sampling Theorem – can be used to derive the relation

$$E\left[\phi^z(Y^\circ_{\nu_n+1})\left[\frac{z}{\lambda^z}\right]^{\nu_{n+1}-\nu_n}\right] = E[z(Q\phi^z)(Y^\circ_{\nu_n})] \qquad n = 1,2,\ldots (7.38)$$

Again, this relation extends to the other eigenvalues provided certain non-degeneracy conditions hold, i.e.,

$$E\left[\phi_i^z(Y^\circ_{\nu_n+1})\left[\frac{z}{\lambda_i^z}\right]^{\nu_{n+1}-\nu_n}\right] = E[z(Q\phi_i^z)(Y^\circ_{\nu_n})] \qquad n = 1,2,\ldots (7.39)$$

for all $1 \le i \le L$.

In analogy with Theorem 5.4, set

$$f(j,u) := E[u^{\nu_{n+1}-\nu_n} I[Y^\circ_{\nu_n+1} = k]], \quad 1 \le j \le L \qquad n = 1,2,\ldots (7.40)$$

for all u in $\mathbb{R}$ with $0 < u \le 1$ and observe that (7.39) can be rewritten as

$$\sum_{j=1}^{L} f(j,u)\phi_i^{z_i(u)}(j) = z_i(u)\sum_{j=1}^{L} f(j,1)\left(Q\phi_i^{z_i(u)}\right)(j), \quad 1 \le i \le L \qquad (7.41)$$

where the notation of Lemma 5.3 has been used. The evaluation of (7.40) follows the same lines as in Theorem 5.4.

(ii): Non-singularity of the system (7.37): It is natural to wonder whether the rank of the system (7.37) is sufficient to determine the scalars $g_n(j,z)$, $1 \le j \le L$, unambiguously. The following lemma provides a simple sufficient condition for this to happen.

Lemma 7.8. *If the infinitesimal generator $\mathcal{I}$ of the continuous-time Markov process $\{Y(t),\ t \ge 0\}$ describing the environment process is diagonalizable, and if the Laplace transform S^* has no finite zero in the right half complex plane, then there exists a real number z_0 in $(0,1)$ such that the linear system (7.37) has a unique solution in a real neighborhood of z_0.*

Proof. In view of Lemma 5.2, it is plain that the condition on S^* implies that all the eigenvectors of T^z are non-zero, and that $|\frac{z}{\lambda_i^z}| < 1$, $1 \le j \le L$, for z in a complex neighborhood of 0. As a result, (7.37) holds for all $1 \le i \le L$ and at least for all z in a right neighborhood of 0.

With the notation of Section 6.2, it is easily seen that this infinitesimal generator $\mathcal{I}$ is given by

$$\mathcal{I} = M(P - I) = H^1. \tag{7.42}$$

The assumption that $\mathcal{I}$ is diagonalizable implies that its eigenvectors, namely ϕ_i^1, $1 \le i \le L$, form a basis of $\mathbb{C}^{L\times 1}$. Hence the vectors ϕ_i^z, $1 \le i \le L$, form a basis of $\mathbb{C}^{L\times 1}$ for all z in $\mathbb{C}$ such that $|z| < z^*$, but for isolated singularities. Indeed, by construction, the coordinates $\phi_i^z(j)$, $1 \le i,j \le L$, of the eigenvectors of the matrix H^z are algebraic functions of the parameter z, so that the determinant Δ^z of the matrix U^z given by

$$U^z := \begin{pmatrix} \phi_1^z(1) & \phi_2^z(1) & \cdots & \phi_L^z(1) \\ \phi_1^z(2) & \phi_2^z(2) & \cdots & \phi_L^z(2) \\ \vdots & \vdots & \ldots & \vdots \\ \phi_1^z(L-1) & \phi_2^z(L-1) & \cdots & \phi_L^z(L-1) \\ \phi_1^z(L) & \phi_2^z(L) & \cdots & \phi_L^z(L) \end{pmatrix} \tag{7.43}$$

is also an algebraic function of the variable z. Since this determinant does not vanish for $z = 1$, its zeros can then only be isolated singularities. This establishes that T^z admits a proper basis for all z in $\mathbb{C}$ with $|z| < z^*$ but for a finite number of isolated singularities.

It is now immediate that there exists a real number $0 < z_0 < 1$ such that for all z in a real neighborhood z_0, the relations $|\frac{z}{\lambda_i^z}| < 1$, $1 \le j \le L$, hold and the rank of the system (7.37) is exactly L. □

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Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique